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Abstract

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JEL: C7

Keywords: Market network system, trading network, core networks, efficient networks

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Abstract

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1 Introduction

It is well known that connections play an important role in the outcomes of many social and economic environments. Not only does an agent's welfare may depend on the connections she or he has with other agents (what we call in social network theory her or his immediate neighbors), but it would also be affected by the connections these agents may have with yet another set of agents. These common sense network effects have lead to the development of study of networks in social sciences. Although the empirical studies of social and economic networks goes back for decades (for example see Katz and Lazarfeild (1955) and Coleman (1966), among others), much of its theoretical development is more recent. Applications of network analysis now range from social issues such friendship relations and marriages (see Wellman and Berkowitz (1988) and Moore (1990)) to economic problems. Economic applications include competition (see Katz and Shapiro (1990)), internal organization of firms (see Marschak and Reichelstein (1993)), cost allocation problem (Hanriet and Moulin (1995)), and transportation networks (see Hendricks et al. (1995)), information and communication networks (for example see Bala and Goyal (2000), Dutta and Jackson (2000), and Bolch and Dutta (2009)), and bargaining (see Corominas-Bolch (2004)), among others.

Since the inception of network analysis, especially in economics, network games have been one of the theoretical foundation of network studies and applications. Ever since the seminal work of Myerson (1977), numerous influential studies have used game theoretic frameworks and solution concepts to study networks. For example, see Myerson (1991), Dutta and Mutuswami (1997), Bala and Goyal (2000), Jackson and Nouweland (2005), and Galeotti (2010), among others. However, one particularly important class of games in which network effects is very much relevant and has been overlooked in the literature is market games which we will consider in this paper.

The study of market games has been the interest of pioneers in game theory. For example, see Shapley (1955), Shubik (1959), Aumann (1964), and Shapley and Shubik (1969a). It is needless to emphasize the importance of market formation and market relations in economics since such issues are at the very foundation of modern economics. The line of research on market games has continued to this day both in cooperative and non-cooperative game theory (see Khan et al. (1997), and Greenberg et al. (2002)). The core solution concept has always been central in much of these studies on market games.

Notably, the importance of connectedness in a market or market relations and therefore the notion of *market network* is essential in not only the *structure* of such market relation formation but also to the extent in which each agent benefits form such market relations. A prime example of this is international trade relations. For example, in formation of a free trade agreements between Country A and Country B it is also important whether or not Country A and/or Country B has a free trade agreement with Country C. It is this kind of motivation that has lead us to formulate *market network system*. Our formulation and analysis of market network systems are important not only in economics but also in other social sciences. It can be applied to a range of economic issues such as the analysis of interpersonal economic relationships, international trading relations as well as formation of preferential trade agreements, and information sharing among economic agents. Applications in other social sciences are numerous. For example, it could be applied to the formation of international security and military agreements, friendship and marital relationships, and neighborhood clustering and segregation or integration (racial-ethnical or economical).

We pay particular attention to the architecture of market networks. We use the notion of core, as well as efficiency, as our solution concepts and study the conditions under which *complete* and *star* networks appear as core networks.[1] Although other forms of network structure are also important, our attention to these two specific types of network structure stems from their significance in economics, especially in international economics. For instance, with regard to preferential trade agreements a complete network structure arises under a global preferential trade agreement, a type of trade arrangement advocated by the World Trade Organization. On the other hand, a star network identifies a hub-and-spoke type of preferential trade relations.

We show that in the absence of any trading costs a complete network appears as a core network. We also show that allocations supported by core networks are *type-based egalitarian allocations* if there is no trading costs and number of traders of each type is equal.[2] Although trading cost has been falling over the past few decades, it is nevertheless costly to trade. In addition to costly trading, the trading technology may not be identical across agents and this in turn leads to trading cost differences between agent. For this reason, we modify our market network system to allow for existence of trading costs as well as heterogeneity in trading technology. We show that for this modified market network system a star network where "the trader with the least trading cost" is the *central trader* is the unique core network.[3] We also show by means of a counter example that allocations supported by core networks does not have type-based egalitarian property even if we have an equal number of traders of each type. Moreover, we prove that all core networks are star networks assuming that trading technology is identical and exhibits economies of scale. Here, we also demonstrate that allocations supported by core networks are not necessarily type-based egalitarian allocations.

We introduce our framework, define market network system and other preliminaries following this introduction in Section 2. Section 3 deals with core networks and presents some results in absence of trading costs. In Section 4, we introduce trading technology (both heterogeneous and identical) and accordingly re-define our market network system and revisit our results of the preceding section regarding the structure of efficient and core networks. We conclude the paper in the last section.

[1]Core is one of oldest solution concepts in economics and game theorists has been interested in studying core for decades. For example, see Shapley and Shubik (1969b) and Shapley (1971)). Jackson and Nouweland (2005) also used core in network theory.

[2]An allocation is type-based egalitarian if traders of the same type benefit equally from a network.

[3]The notion of central trader is analogous to the notion of middle man in Kalai et al. ((1978).

2 Preliminaries

Let $N = \{1, 2, 3, ..., n\}$ be the finite set of traders. Assume that there are m commodities in an economy. Let $\omega^i \in \mathbb{R}^m_+$ be the endowment of trade $i \in N$. Similarly, denote the endowment of any subset of traders $S \subset N$ by $\omega^S \in \mathbb{R}^m_+$. A network g is a list of unordered pairs of traders. More formally, let G^N be the set of all subsets of N with cardinality of 2. Similarly, for any $S \subset N$ denote by G^S the set of all subsets of S with size 2. Then the set of all networks on N is given by $G = \{g | g \subset G^N\}$. Elements of any $g \in G$ are trade links between pairs of traders. For notational simplicity we denote a trade link between trader $i, j \in N$ by ij. The empty set represents the cases where there is no link between any two traders. For any network $g \in G$ and any subset of traders $S \subset N, g(S) = g \cap G^S$ is the subnetwork of g on the set S, a restriction of g on S. A path in network $g \in G$ between traders i and j, denoted by $\pi_g(ij)$, is a sequence of linked traders $i_1, i_2, ..., i_K$ such that $i_k i_{k+1} \in g, \forall k \in \{1, 2, ..., K-1\}$ where $i_1 = i$ and $i_K = j$. Denote by $\Pi_g(ij)$ the set of all paths between traders i and j in network g. Similarly, let $\Pi_{g(S)}(ij), S \subset N$, be the set of all paths between traders i and j in subnetwork $g(S) \subset g$.

A component of a network $g \in G$ is a subnetwork $g(S) \subset g, S \subset N$, where $i, j \in S$ if and only if there exists a path $\pi_g(ij) \in \Pi_g(ij)$ That is, a component is a subnetwork where there is a path between any two pair of traders in this subnetwork and if there is path between any pair of traders then they must belong to the same subnetwork. For any network $g \in G$, denote the set of its components by $C(g)$, i.e., $C(g) = \{g(S) \mid \forall i, j \in S \iff \exists \pi(ij) \in \Pi_g(ij)\}$. A network $g \in G$ is connected if $\Pi_g(ij) \neq \emptyset, \forall i, j \in N$. A network $g \in G$ is a complete network if $\forall i, j \in N$, $ij \in g$. A network is said to be a star network with trader $i \in N$ as its central player, denoted by g_i^s, if $ij \in g_i^s, \forall j \in N$ and $jk \notin g_i^s$, for all $j, k \in N$ where $j \neq i$ and $k \neq i$. That is a network is a star network with trader i as its central player if every other traders is linked with i but they are not linked with each other.

Define a utility function $u : \mathbb{R}^m_+ \mapsto \mathbb{R}_+$. We assume throughout the rest of the paper that u is monotonic, concave, and homogeneous of degree 1.[4] In general, a value function is $V : G \mapsto \mathbb{R}$. We define the value function for a market network as $V(g) = \sum_{g(S) \in C(g)} V(g(s)), \forall g \in G$ where $V(g(s)) = u(\omega^S)$. That is, the value function is component-additive. We say a value function

[4] u is monotonic if $u(y) > u(x)$, where $y \geq x$ $\forall x, y \in \mathbb{R}^m_+$. u is concave if $\forall x, y \in \mathbb{R}^m_+, u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$, where $\lambda \in (0, 1)$. u is homogeneous of degree r if $u(\mu x) = \mu^r u(x), \mu > 0$.

$V(g), g \in G$ is component-convex if for any $g \in G, \forall g(S) \in C(g)$ and $\forall g(T_1) \subset g(S), g(T_2) \subset g(S)$ where $T_1, T_2 \subset S$ and $T_1 \cap T_2 = \emptyset$ we have $V(g(S)) > V(g(T_1) + V(g(T_2))$. A market network system is a quadruple $\Psi \equiv (N, G, \omega^N, V)$.

Lemma 1. *Consider a market network system $\Psi \equiv (N, G, \omega^N, V)$. Value function V is component-convex if u is concave, monotonic and homogeneous of degree $r \geq 1$.*

A feasible allocation under network $g \in G$ is defined as $\nu \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \nu^i \leq V(g)$. Denote the set of all feasible allocations for network $g \in G$ by $A|g$, i.e., $A|g = \{\nu \in \mathbb{R}^n_+ | \sum_{i \in N} \nu_i \leq V(g)\}$. We also denote for any $g \in G$ a proposed allocation $\nu \in A|g$ by $\nu|g$. Let Ω be the set of all allocation-network pairs, i.e., $\Omega = \{\nu|g \quad | g \in G, \quad \nu \in A|g\}$. Consider networks $g, g' \in G$. We say that g' is reachable from g via $S \subset N$, denoted by $g \longmapsto_S g'$ if:

1) $ij \in g' \setminus g$, then $i, j \in S$; and

2)$ij \in g \setminus g'$, then either a) $i \in S$ and $j \in N \setminus S$ or b) $i, j \in S$.

3 Core Networks

A market network $g' \in G$ is an efficient network for a system $\Psi \equiv (N, G, \omega^N, V)$ if $V(g') \geq V(g), \forall g \in G$. Similarly, we say an allocation network $\nu|g \in \Omega$ is efficient if g is efficient and $\sum_{i \in N} \nu_i = V(g)$. That is, a pair of allocation-network is efficient if the network is an efficient network and the allocation is a Pareto efficient given that network. Denote the set of all efficient allocation-network pairs for market network system Ψ by $\mathbb{E}(\Psi) \subset \Omega$. An allocation-network pair $\nu'|g' \in \Omega$ dominates another pair of allocation-network $\nu|g \in \Omega$ via $S \subset N$, denoted by $\nu|g \prec_S \nu'|g'$ if $g \longmapsto_S g'$ and there exists a component $g'(S) \in C(g')$ such that $\nu'_i > \nu_i \forall i \in S$. Denote the set of allocation-network pairs that dominate an allocation-network pair $\nu|g$ by $\Delta(\nu|g)$. A pair of allocation-network $\nu|g \in \Omega$ is a core allocation-network if $\Delta(\nu|g) = \emptyset$. Denote the set of all core allocation-network pairs for a market network system Ψ by $\mathbb{C}(\Psi)$, that is, $\mathbb{C}(\Psi) = \{\nu|g \in \Omega \mid \Delta(\nu|g) = \emptyset\}$.

Claim 1. *Assume any market network system $\Psi \equiv (N, G, \omega^N, V)$. Then, $\mathbb{C}(\Psi) \subset \mathbb{E}(\Psi)$.*

The following theorem addresses the architecture of market networks. Particularly, it highlights the condition under which a complete network appears as a core network. Complete networks are the most important type of network structures with a nice inclusiveness property. Among the most appealing applications of complete market networks is the notion of a global free trade agreement, which is advocated by World Trade Organization.

Theorem 1. *Consider any market network system* $\Psi \equiv (N, G, \omega^N, V)$ *and assume that* V *is component-convex. Then the complete network* g_c *is supported by a pair* $\nu|g_c \in \mathbb{C}(\Psi)$*.*

It follows from Claim 1 and Theorem 1 that the complete network is an efficient network as they indicate that there exists a pair of allocation-network $\nu|g_c \in \mathbb{E}(\Psi)$. Next we further characterize core networks.

Theorem 2. *Consider any market network system* $\Psi \equiv (N, G, \omega^N, V)$ *and assume that* V *is component-convex. If* $\nu|g \in \mathbb{C}(\Psi)$*, then* g *is a connected network.*

For every trader $i \in N$ define his type by the size of his endowment. Let $H = \{1, 2, 3, ..., h\}$ be the set of types. We can then re-index the set of traders N by a set that identifies their types and defined as $NH = \{11, 12, 13, ..., 1n_1, 21, 22, 23, ..., 2n_2, ..., h1, h2, h3, ..., hn_h\}$ where $n_i, i \in H$ denotes the number traders of type i. Clearly, we have $|NH| = \Sigma_{i \in H} n_i = n$. To maintain mathematical simplicity and tractability we assume in the rest of the paper that $n_i = \bar{n}, \forall i \in H$, that is, we have equal number of traders of each type. We denote the endowment of trader $ij \in NH$ (trader j of type i) by ω^{ij}.

The notion of equity has been a focal and perhaps a controversial issue since the dawn of human civilization. It has also been a subject of scholarly debates since Plato. Here, we present a notion of equity based on players' type. We say that an allocation $\nu \in A|g, g \in G$, is type-based egalitarian if $\nu^{ij} = \nu^{ik}, \forall ij, ik \in NH$. That is, an allocation is type-based egalitarian if traders of the same type benefit equally from a network. The following theorem addresses this important property of a core pair of allocation-network.

Theorem 3. *Consider any market network system* $\Psi \equiv (N, G, \omega^N, V)$ *and assume that there is an equal number of traders of each type* $i \in H$*. Then, if* $\nu|g \in \mathbb{C}(\Psi)$*, then* ν *is a type-based egalitarian allocation.*

It is noteworthy that the assumption of equal number of traders of each type is crucial in egalitarian property we presented in Theorem 3. The following example highlights this point.

Example 1:

Assume $N = \{1, 2, 3\}$, $m = 2$, and $u = x_1^{0.5} x_2^{0.5}$ where x_1 and x_2 are quantities of good 1 and 2, respectively. Let also the endowments of traders be $\omega^1 = (1, 0), \omega^2 = \omega^3 = (0, 0.5)$. Then the value function V can be defined as:

$V(\{12, 13, 23\}) = V(\{12, 13\}) = V(\{21, 23\}) = V(\{31, 32\}) = 1$

$V(\{1i|i = 2, 3\}) = 0.7$

$V(\{23\}) = V(\emptyset) = 0.$

Now, consider allocation-network pair $\nu|g^c = (0.7, 0.3, 0.0)|\{12, 13, 23\}$. Clearly, $\nu|g^c \in \mathbb{C}(\Psi)$ but allocation ν is not a type-egalitarian allocation.

4 Heterogeneous Trading Technology

In the preceding section we assumed that there is no trading cost. This zero-trading cost assumption implicitly assumes homogeneity of trading technology across all traders although we have allowed for endowment heterogeneity across traders. Even though such type of endowment heterogeneity is sufficient in some environments, it is also crucial to formulate a scenario where trader technologies are not identical. We shall now consider such a possibility.

For any network $g \in G$ and for any $i \in N$ define by $g_i \subset g$ the set of links that player i has at network g. Let a trading technology for any trader $i \in N$ be represented by a set-valued cost function $\tau^i : \mathbb{Z}_+ \mapsto\mapsto \mathbb{R}^m_+$ where $\mathbb{Z}_+$ is the set of non-negative integers. We also denote this trading cost for every player $i \in N$ at network $g \in G$ as $\tau^i(|g_i|)$. This cost function states that the trading costs for traders are measured in real terms and take the iceberg type. In other words, given any network, traders pay their trading costs in the form of commodities and depend only on the number links they have to make at any network. We assume that $\tau^i(|g_i|)$ is monotonically increasing in $|g_i|, \forall g \in G, \forall i \in N$. We further assume that this cost function is increasing in the in the index of traders, i.e., $\tau^1 < \tau^2 < \tau^3 < .... < \tau^n$. Finally, without loss of generality let $\tau^1 = 0$. To simplify further and make our analysis more tractable, we assume that for any network $g \in G$ and for any link $ij \in g$ the cost of forming the link between the two traders is equally shared between $i, j \in N$ and that such costs will be the minimum of the two trading costs, i.e., $min\{\tau^i, \tau^j\}$. Then, we redefine a value function as: $V(g) = \Sigma_{g(S) \in C(g)} V(g(S)), g \in G, S \subset N$

where $V(g(S)) = \max\{0, u(\omega^S - \Sigma_{ij \in g(S)} min\{\tau^i, \tau^j\})\}$. It is noteworthy that our value function is no longer component-convex due to these trading costs. Recall that we denoted by $g_i^s \in G$ a star network with player $i \in N$ as its central player. The following claim states a nice property of networks that are either include or included in a star network. Apart from its intrinsic value, this claim will be useful in obtaining our proceeding results.

Claim 2. *Consider any nonempty network $g \in G$. If $g \subset g_i^s$ or $g_i^s \subset g$ for any $i \in N$, then g is a connected network.*

Theorem 4. *Let $\Psi \equiv (N, G, \omega^N, V)$ be a market network system where τ is strictly increasing in index of traders. Then, $\nu|g_1^s \in \mathbb{E}(\Psi)$ where $\nu \in A|g_1^s$ is any efficient allocation. Moreover, g_1^s is a unique efficient network.*

We next characterize the structure of core networks for our market network system. First, in the result that follows, we formally state a condition that guarantees that a star network will be a unique core network.

Theorem 5. *Let $\Psi \equiv (N, G, \omega^N, V)$ be a market network system where τ is strictly increasing in index of traders. Then, for any $\nu|g \in \mathbb{C}(\Psi)$, g is unique and defined as $g = g_1^s$.*

Next, it is interesting to study whether type-based egalitarian allocations will be supported by core networks as in Theorem 3 of the preceding section. Here, we have to redefine type since endowment alone cannot be determinant of a trader's type. In the context of the current section, we say that any two traders $i, j \in N, i \neq j$ are of the same type if $\omega^i = \omega^j$ and $\tau^i(z) = \tau^j(z), \forall z \in \mathbb{Z}_+$. That is, traders are of the same type if their endowment and trading technology are identical. We claim that type-based egalitarian property of allocations supported by core networks are no longer valid even if the conditions of Theorem 3 are met. We show this by a counter example. However, for the sake of efficiency of our presentation, we postpone it till the end of this section (see Example 2).

Recall that we denoted the set of link that any trader $i \in N$ has in network $g \in G$ by g_i. We say that a trading technology for trader $i \in N$ exhibits increasing returns to scale if $\bar{\tau}^i \equiv \tau^i/|g_i|$ is decreasing in $|g_i|$ for all $g \in G$. Next assume that all traders have the same trading technology. The following result shows the kind of network structures that appear as a core network.

Theorem 6. *Consider a market network system $\Psi \equiv (N, G, \omega^N, V)$ and assume that all traders have an identical trading technology that exhibits economies of scale. Then, for all $\nu|g \in \mathbb{E}(\Psi)$, g is connected.*

Corollary 1. *Consider a market network system $\Psi \equiv (N, G, \omega^N, V)$ and assume that all traders have an identical trading technology that exhibits economies of scale. Then, for all $\nu|g \in \mathbb{C}(\Psi)$, g is connected.*

Theorem 7. *Consider a market network system $\Psi \equiv (N, G, \omega^N, V)$ and assume that all traders have an identical trading technology that exhibits economies of scale. Then, for all $\nu|g \in \mathbb{C}(\Psi)$, g is a star network and defines as $g \equiv g_i^s$ for any $i \in N$.*

Here, as stated earlier, type-based egalitarian allocation will not be supported by core networks even if the conditions of Theorem 3 are met. That is, contrary to the results of Theorem 3, type-based egalitarian allocations may not be supported by core networks even though we have an equal number of traders of the same type. The following is an example that makes this point.

Example 2:

Assume $N = \{1, 2, 3, 4\}$, $m = 2$, and $u = x_1^{0.5} x_2^{0.5}$ where x_1 and x_2 are quantities of good 1 and 2, respectively. Let also the endowments of traders be $\omega^1 = \omega^2 = (1, 0), \omega^3 = \omega^4 = (0, 1)$. Finally, assume that the trading technology is identical and given $\forall i \in N$ and $\forall g \in G$by:

$$\tau^i(|g_i|) = \begin{cases} [0.2 + 0.1|g_i|]\mathbf{J}^T & |g_i| > 0 \\ 0 & |g_i| = 0 \end{cases}$$

where $\mathbf{J} = [1 \quad 1]$ is a unit matrix with a dimension of 1×2, therefore τ is a 2×1 vector. Then the value function V can be defined as:

$V(g_i^s) = 1.5$

$V(g^c) = 0.8$

$V(\{ij, jk, km, mi\} = 1.2 \quad \forall i, j, k, m \in N$

$V(\{ij, jk, km\} = 1.3 \quad \forall i, j, k, m \in N$

$V(\{ij, ik\} = 0.77 \quad \forall i, j, k \in N$

$V(\{13, 24\} = V(\{14, 23\} = 1.4$

$V(\{13\} = V(\{14\}) = V(\{23\}) = V(\{24\} = 0.7$

$V(g) = 0 \quad otherwise$

It is noteworthy that networks with a structure such as $\{ij, jk, km, mi\}$ where $i, j, k, m \in N$ are called circles while network structures such as $\{ij, jk, km\}$ where $i, j, k, m \in N$, are called line networks. Clearly in this example we have equal number of traders of each type. However, allocation-network $(0.45, 0.35, 0.35, 0.35)|g_1^s \in \mathbb{C}(\Psi)$ while traders 1 and 2 are of the same type.

5 Conclusion

Network economics has recently become a well-received tool of economic analysis. Its wide applications range across most fields of economics, including industrial organization, information economics, transportation economics, and international economics. Game theoretic approach to the analysis of networks has also been a main pathway in studying networks since the seminal work of Myerson (1977). Despite the tremendous attention and growth that network analysis and related application have experienced over the past decade, one very fundamental class of network games has been overlooked in the literature on which this paper has focused. Parallel to the notion of market game, we introduced the notion of market network system.

Market formation and its outcomes are clearly corner stones of economic analysis. As connections and the ability of agents to make such connections are indispensable for market formation and its outcome, the market network is very fundamental in our understanding of markets. This has motivated us to formulate the notion of market network system. We then studied the structure of market networks using the notion of efficient network and core network. We show that a complete network, with an type-based egalitarian allocation of network value, is in the set of core allocations-networks if there is no trading cost. We then introduced agent heterogeneity by introducing heterogeneous trading technologies represented by a cost correspondence. Given this networking cost heterogeneity, we demonstrated that a star network with the least cost agent as its central trader is the *unique* core network.

Our analysis of network can be used in a variety applications in economics. For example, it can be used in international economics to study the formation of preferential trade agreements. As another example, our model can be used in analyzing trading in financial markets. It is also interesting to apply our notion of market network system and core network outcomes in information market. In addition to these examples of possible economic application, our result can be applied in other social and political contexts such as international security agreements and friendship and marital relations.

Appendix: Proofs

Proof of Lemma 1. Let u be concave, monotonic and homogeneous of degree $r \geq 1$. Fix any $g \in G$ and any $g(S) \in C(g)$. Consider any $g(T_1), g(T_2) \subset g(S)$ such that $T_1 \cap T_2 = \emptyset$. Without loss of generality let both $g(T_1)$ and $g(T_2)$ be connected subnetworks. On the one hand, we have $V(g(S)) = u(\omega^S)$, recalling that $g(S)$ is a component. On the other hand, we have $V(g(T_1)) = u(\omega^{T_1})$ and $V(g(T_2)) = u(\omega^{T_2})$. Due to concavity assumption we have $u(\omega^{T_1}/2+\omega^{T_2}/2) > u(\omega^{T_1})/2+u(\omega^{T_2})/2$. Then, by homogeneity assumption we have $u(\omega^{T_1} + \omega^{T_2}) = 2^r u(\omega^{T_1}/2 + \omega^{T_2}/2)$. Thus, from our homogeneity and concavity assumptions combined, we conclude that $u(\omega^{T_1}+\omega^{T_2}) > 2^r[u(\omega^{T_1})/2 + u(\omega^{T_2})/2] > u(\omega^{T_1}) + u(\omega^{T_2})$. Since $T_1, T_2 \subset S$, we have $\omega^S \geq \omega^{T_1} + \omega^{T_2}$. It then follows from monotonicity of u that $u(\omega^S) \geq u(\omega^{T_1}+\omega^{T_2})$, which in turn implies that $u(\omega^S) > u(\omega^{T_1}) + u(\omega^{T_2})$. Therefore, $V(g(S)) > V(g(T_1)) + V(g(T_2))$, that is, V is component-convex. □

Proof of Claim 1. Consider the system $\Psi \equiv (N, G, \omega^N, V)$ and assume the negation, that is, $\exists \nu | g \in \mathbb{C}(\psi) \setminus \mathbb{E}(\Psi)$. This implies that there exists $\hat{g} \in G$ such that $V(\hat{g}) > V(g)$. Fix an allocation $\hat{\nu}^i = \nu^i + (V(\hat{g}) - V(g))/|N|$. Since $V(\hat{g}) > V(g)$, $\hat{\nu}^i > \nu^i, \forall i \in N$. Clearly, $g \longmapsto_N \hat{g}$ since the grand coalition of traders can reach any network they desire. Moreover, we also have $\hat{\nu} \in A|\hat{g}$ since $\Sigma_{i\in N}\hat{\nu}^i = \Sigma_{i\in N}\nu^i + \Sigma_{i\in N}(V(\hat{g}) - V(g))/|N| \leq V(\hat{g})$, where the last inequality follows from the fact $\nu \in A|g$. All these imply that $\hat{\nu}|\hat{g} \in \Delta(\nu|g)$ implying that $\nu|g \notin \mathbb{C}(\Psi)$, which is a contradiction. □

Proof of Theorem 1. Assume the negation. That is, for market network system Ψ, there does not exist $\nu|g_c \in \mathbb{C}(\Psi)$ for any $\nu \in A|g_c$, implying $\Delta(\nu|g_c) \neq \emptyset, \forall \nu \in A|g_c$. It then must be the case that $\forall \nu \in A|g_c$ there exist $S \subset N$, $g' \in G$, $g'(S) \in C(g')$, and $\nu' \in A|g'$ such that $g_c \longmapsto_S g'$ $\nu^S \in A|g'(S)$ and $\nu'^i > \nu^i, \forall i \in S$. Since our non-existence assumption stipulates for all $\nu \in A|g_c$, all these imply that $V(g'(S)) > V(g_c)$. Since g_c is a complete network, we must have $g' \subset g_c$. Now consider two possibilities: Case 1) $|C(g')| = 1$, i.e., g' has a single nontrivial (nonempty) component. This implies that $V(g_c) = u(\omega^N) \geq u(\omega^S) = V(g'), S \subset N$, where the inequality follows from the monotonicity of u since $\omega^N \geq \omega^S$. This is a contradiction since we showed that our negation assumption implies $V(g'(S)) > V(g_c)$. Case 2) $|C(g')| > 1$. Without loss of generality, let $|C(g')| = 2$. Denote these components by $g(S)$ and $g(T)$. It then follows from the definition of

V and our negation assumption that $V(g') = V(g(S)) + V(g(T)) > V(g_c)$. However, this inequality contradicts our component-convexity assumption. $\square$

Proof of Theorem 2. Assume the negation, i.e., let g not be connected for $\nu|g \in \mathbb{C}(\Psi)$. Then, we must have $|C(g)| > 1$. Without loss of generality, assume that $|C(g)| = 2$. Denote the components of g by $g(S)$ and $g(T)$, $S, T \subset N$. Now construct any single-component network $g' \in G$ where $g' = \{ij\} \cup g(S) \cup g(T)$ where $i \in S$ and $j \in T$. By component-convexity of V, we must have $V(g') > V(g)$. Next, construct an allocation ν' such that $\nu'^i = \nu^i + (V(g') - V(g))/|N|, \forall i \in N$. Clearly, by construction $\nu'^i > \nu^i, \forall i \in N$ and $\sum_{i \in N} \nu'^i = \sum_{i \in N} \nu^i + V(g') - V(g)$. However, it follows from Claim 1 that $\nu|g \in \mathbb{E}(\Psi)$, implying that $\sum_{i \in N} \nu^i = V(g)$. This, in turn implies that $\Sigma_{i \in N} \nu'^i = V(g')$, that is, $\nu' \in A|g'$. Moreover, clearly the grand coalition of traders can induce any network from any other network, i.e. $g \longmapsto_N g'$. All these indicate $\nu|g \prec_N \nu'|g'$. Thus, we have $\nu'|g' \in \Delta(\nu|g)$, implying that $\nu|g \in \Omega \setminus \mathbb{C}(\Psi)$ which is a contradiction. $\square$

Proof of Theorem 3. Assume the negation. Without loss of generality assume that $\nu^{i1}, \forall i \in H$, is the lowest allocation for each type. Construct a coalition of traders $S = \{11, 21, 31, ..., h1\}$. Note that by construction, we have $\omega^S = (1/\bar{n})\omega^N \equiv (1/\bar{n})\omega^{HN}$. Now consider $g' \in G$ such that $g'(S) \in C(g')$ and formed by traders in S severing their possible links with all traders $j \in N \setminus S$ under network g and form any link among themselves to guarantee that $g'(S)$ is connected if $g(S)$ is not connected (this implies that $g \longmapsto_S g'$). It then follows that $V(g'(S)) = u(\omega^S) = (1/\bar{n})u(\omega^N) = (1/\bar{n})V(g)$, where the second equality follows from linear homogeneity of u and the last equality is due to Theorem 2 which implies that a core network must be a single-component network. On the other hand, we must by construction have $\Sigma_{i \in H} \nu^{i1} < \Sigma_{i \in H} \bar{\nu}^i$ where $\bar{\nu}^i = (1/\bar{n})\Sigma_j^{\bar{n}} \nu^{ij}$ is the initial average allocation for each type. Next, construct an allocation ν' such that $\nu'^{i1} = (1/h)V(g'(S)), \forall i \in S$. Since $|S| = |H| = h$, it follows that $\nu'^S \in A|g'(S)$. In addition, also by construction, $(1/h)V(g'(S)) = (1/h\bar{n})V(g) = (1/h\bar{n})\Sigma_{i=1}^h \Sigma_{j=1}^{\bar{n}} \nu^{ij} = (1/h)\Sigma_{i=1}^h \bar{\nu}^i > \nu^{i1}, \forall i \in h$. Note that the second equality follows again from Claim 1 and Theorem 2 as they imply the Pareto efficiency of ν and the inequality is due to our construction. All in all, these indicate that there exist $g' \in G, S \subset N, g \longmapsto_S g', g'(S) \in C(g')$ and $\nu'^S \in A|g'(S)$ such that $\nu^i < \nu'^i, \forall i \in S$. Thus, we have $\nu'|g' \in \Delta(\nu|g)$, contradicting that $\nu|g \in \mathbb{C}(\Psi)$. $\square$

Proof of Claim 2. First note that any star network is connected by its definition. Then, clearly if $g \subset g_i^s, i \in N$ and $g \in G$ then g is also connected unless g is an empty network. Now, let $g_i^s \subset g$, $i \in N$. Then, by the definition of g_i^s, we have $ij \in g_i^s$ for all $j \in N, i \neq j$. It follows that $\Pi_g(jk) \neq \emptyset$ for all $j, k \in N$. Thus, g is connected. □

Proof of Theorem 4. First we show that $\nu|g_1^s \in \mathbb{E}(\Psi)$ for any $\nu \in A|g_1^s$ such that $\Sigma_{i\in N}\nu^i = V(g_1^s)$. To show this, it suffices to verify that $V(g_1^s) \geq g, \forall g \in G$. To prove this, consider three possibilities:

1) All $g \in G$ such that $g \subset g_1^s$: Since g_1^s is a connected network by definition, g must also be a connected network due to Claim 2. This implies that $|C(g_1^s)| = |C(g)| = 1$. Thus, we have $V(g_1^s) = u(\omega^N) > u(\omega^S) = V(g(S)) = V(g)$ where $g(S) \in C(g)$. The inequality follows from monotonicity of u.

2) All $g \in G$ such that $g_1^s \subset g$: Again, it follows from Claim 2 that g is a connected network. As a result, and similar to the preceding case, both networks are single-component networks. Therefore, we have $V(g_1^s) = u(\omega^N)$ and $V(g) = u(\omega^N - \Sigma_{ij\in g\setminus g_1^s} min\{\tau^i, \tau^j\})$. Since $\tau^i > \tau^1 = 0$ for all $i \in N \setminus \{1\}$ due to our assumption on τ, we have $\Sigma_{ij\in g\setminus g_1^s} min\{\tau^i, \tau^j\} > 0$. This implies that $V(g_1^s) > V(g)$ because of monotonicity of u.

3) Consider any $g \in G$ such that both $g \not\subset g_1^s$ and $g_1^s \not\subset g$: It follows that there must exist $ij \in g \setminus g_1^s$. Define $g(T) \subset g$ such that $g(T) = \{ij \mid ij \in g \setminus g_1^s\}$. There are two sub-cases: 3a) $g(T) \notin C(g)$, implying that g is connected and $\exists 1k \in g$ for some $k \in N \setminus \{1\}$. Next, construct a network $g' \in G$ by all traders with any link $ij \in g, i, j \neq 1$, severing their links and forming a link with traders 1. Clearly, $g' \subset g_1^s$. Then, on the one hand, we have $V(g_1^s) > V(g')$ by Case 1 above. On the other hand, we have $V(g') = u(\omega^S) > u(\omega^S - \Sigma_{ij\in g(T)} min\{\tau^i, \tau^j\})$ where $g(S) \in C(g)$. Recall that we already established that $C(g)$ is singleton and note that the inequality is due to monotonicity of u and the fact that $\Sigma_{ij\in g(T)} min\{\tau^i, \tau^j\} > 0$. It then follows that $V(g_1^s) > V(g') > V(g)$. 3b) Let $g(T) \in C(g)$. Without loss of generality, let $|C(g)| = 2$ and assume that $g(N \setminus T) \in C(g)$. Then, $V(g) = V(g(N\setminus T)) + V(g(T)) = u(\omega^{N\setminus T}) + u(\omega^T - \Sigma_{ij\in g(T)} min\{\tau^i, \tau^j\})$. Recall that $g(N\setminus T) \subset g_1^s$ which implies $1 \in N \setminus T$ and that $\Sigma_{ij\in g(T)} min\{\tau^i, \tau^j\} > 0$ by our assumption on τ. Now construct $g' \in G$ such that $g'(T') \equiv g_1^s(T')$ where $T' = T \cup \{1\}$, implying that $g' \equiv g_1^s$. It then follows that $V(g') = u(\omega^N) = u(\omega^{N\setminus T} + \omega^T) > u(\omega^{N\setminus T}) + u(\omega^T) > u(\omega^{N\setminus T}) + u(\omega^T - \Sigma_{ij\in g(T)} min\{\tau^i, \tau^j\}) = V(g)$ where the first inequality is due to homogeneity and concavity of u while the second inequality

is due to monotonicity of u.[5]

It is left to show g_1^s is the unique efficient network. Let $g \in G$ be any efficient network. We claim that: 1) $g \subset g_1^s$; 2) $g_1^s \subset g$. In what follows we prove this claim. 1) $g \subset g_1^s$: Assume the negation, i.e., $\exists ij \in g \setminus g_1^s$. Then, by Case 3 above, we have $V(g_1^s) > V(g)$ contradicting that g is an efficient network. This also establishes by Claim 2 that any efficient network should be connected. 2) $g_1^s \subset g$: Assume the negation; that is, $\exists ij \in g_1^s \setminus g$. Since we have already established that any efficient network $g \in G$ is connected, $|C(g)| = 1$. Let $g(S) \in C(g)$ be the single component of g. Then, $V(g) = u(\omega^S - \Sigma_{ij \in g(S)} min\{\tau^i, \tau^j\})$. Since we assumed that $g \neq g_1^s$, $\Sigma_{ij \in g(S)} min\{\tau^i, \tau^j\} > 0$. Since $V(g_1^s) = u(\omega^N), \omega^N \geq \omega^S, \forall S \subset N$, we conclude that $V(g_1^s) > V(g)$ contradicting efficiency of g.

Since we established that $g \subset g_1^s$ and $g_1^s \subset g$, we conclude that $g = g_1^s$ for any efficient network $g \in G$. Thus, g_1^s is the unique efficient network for our system Ψ. □

Proof of Theorem 5. Let $\nu|g \in \mathbb{C}(\Psi)$. It follows from Claim 1 that $\nu|g \in \mathbb{E}(\Psi)$. Theorem 4 indicates that g_1^s is the unique efficient network. All these imply that g_1^s is the unique core network. □

Proof of Theorem 6. Assume the negation, i.e., let $g \in G$ be an efficient network and not be connected. Wlg let $|C(g)| = 2$. Let also $g(S), g(T) \in C(g)$. Pick trader $i \in \arg\max_{i \in N}\{|g_i| \quad |i \in N\}$. Wlg let also $i \in S$. By our assumption of scale economies, we have $\tau^i/|g_i| \leq \tau^j/|g_j|, \forall j \in N$. The value of g is defined as $V(g) = V(g(S)) + V(g(T)) = u(\omega^S - \tau^i(|g_i|) - \Sigma_{kj \in g(S)\setminus\{ij | ij \in g(S), \forall j \in S \setminus \{i\}} \min\{\tau^j, \tau^k\}) + u(\omega^T - \Sigma_{kj \in g(T)} \min\{\tau^j, \tau^k\})$. Now fix a network $g' \in G$ such that all traders $j \in T$ sever their links and each forms a new links with trader i and $g(S) = g'(S)$, i.e., all traders in $j \in S$ maintain their links under g. The value of this network is $V(g') = u(\omega^{S \cup T} - \tau^i(|g'|)) > u(\omega^{S \cup T} - [\tau^i(|g|) + \Sigma_{kj \in g(S)\setminus\{ij|ij \in g(S), \forall j \in S\setminus\{i\}} min\{\tau^j, \tau^k\} + \Sigma_{kj \in g(T)} min\{\tau^j, \tau^k\}]) \equiv u(x+y)$ where $x \equiv \omega^S - [\tau^i(|g|) + \Sigma_{kj \in g(S)\setminus\{ij|ij \in g(S), \forall j \in S\setminus\{i\}} min\{\tau^j, \tau^k\}]$ and $y \equiv \omega^T - \Sigma_{kj \in g(T)} min\{\tau^j, \tau^k\}$. The inequity is due to scale economies of τ^i, the fact that trader i has the lowest cost under g, and monotonicity of u. On the other hand, concavity and homogeneity of u implies that

[5]Recall that assuming homogeneity with a degree of one or greater combined with concavity of u implies super additivity of u since $u(x + y) = 2^r u(x/2 + y/2) > 2^r [u(x)/2 + u(y)/2] \geq u(x) + u(y), \forall x, y \in \mathbb{R}_+^m$, if $r \geq 1$ where the first inequity is because of homogeneity of u, the second inequality is due to concavity of u, and the last one is guaranteed if our condition on the degree of homogeneity r is met.

$u(x+y) > u(x) + u(y) \equiv V(g)$.[6] Therefore, we conclude that $V(g') > V(g)$, contradicting with g being an efficient network. $\square$

Proof of Corollary 1. It directly follows from Claim 1 and Theorem 6. $\square$

Proof of Theorem 7. Assume the negation, i.e, there exists $\nu|g \in \mathbb{C}$ where $g \neq g_i^s, \forall i \in N$. Since a core network is connected, we have $|C(g)| = 1$. Let $i \in \arg\max_{i \in N}\{|g_i| \quad |i \in N\}$, that is trader i is a trader with maximum number of links. Construct a network $g' \in G$ such that $g'(S) = g_i^s(S)$ where $S = \{i\} \cup \{k|ik \notin g, k \in N\}$ and $g'(N \setminus (S \setminus \{i\})) = g(N \setminus (S \setminus \{i\}))$, that is, $g' = g_i^s$. Now, we have $V(g) = u(\omega^N - \Sigma_{kj \in g} \min\{\tau^j, \tau^k\}) = u(\omega^N - \tau^i(|g_i|) - \Sigma_{kj \in g \setminus \{ij | ij \in g, \forall j \in N\}} \min\{\tau^j, \tau^k\})$. Note that by economies of scale assumption and since trader i has the maximum number of links, we have $\tau^i(|g_i|)/|g_i| \leq \tau^j(|g_i|)/|g_j|$ for all $j \in N$. On the other hand, the value of network g' is given by $V(g) = u(\omega^N - \tau^i(|g_i'|)$. Since $|g_i'| > |g_i|$, we conclude again from our economies of scale assumption that $\tau^i(|g_i'|)/|g_i'| < \tau^i(|g_i|)/|g_i|$, which in turn implies that $\tau^i(|g_i'|) < \tau^i(|g_i|) + \Sigma_{kj \in g \setminus \{ij | ij \in g, \forall j \in N\}} \min\{\tau^j, \tau^k\})$. Therefore, by monotonicity of u, we have $V(g') > V(g)$. Next construct an allocation ν' such that $\nu'^j = \nu^j + [V(g') - V(g)]/|N|, \forall j \in N$. Clearly, by construction $\Sigma_{j \in N} \nu'^j = V(g')$ and $\nu'^j > \nu^j, \forall j \in N$. Moreover, we trivially have $g \longrightarrow_N g'$. Thus, we conclude that $\nu|g \prec_N \nu'|g'$, implying that $\nu'|g' \in \Delta(\nu|g)$ which is a contradiction. $\square$

[6] See Footnote 4.

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