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Abstract

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JEL: D8, C7

Keywords: Economic networks, social networks, stability

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On Network Stability Once Again

Abstract

We introduce a notion of network stability which is in the spirit of von Neumann and Morgenstern (1947). We then study the structure of stable economic networks and their associated stable allocations by analyzing the conditions under which complete networks and star networks are stable. We also address conditions for existence and uniqueness of a set of stable networks.

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1 Introduction

Although the analysis of networks goes back to seminal work of Myerson (1977), much of its development has taken place over the past decade or so. After just a decade of intense interest and research, the study of networks has now become a well-documented tool of analysis of interactions among economic and social agents. The applications of network analysis range from social issues such as friendship relations (see, for example, Wellman and Berkowitz (1988)) to economic problems such as communication and information networks (see Bala and Goyal (2000) and Bolch and Dutta (2009)), bargaining (see Corominas-Bosch (2004)), international trade (see Furusawa and Konishi (2007)), cost allocation problem (see Henriet and Moulin (1996)), and transportation networks (see Hendricks et al. (1995)).

Ever since the start of study of social and economic networks, game theoretic solution concepts have been widely used in the context of network analysis. For example, Myerson (1977) and Jackson and Nouweland (2005) studied cooperative solution concepts. On the other hand, numerous authors used non-cooperative solution concepts such as variants of Nash equilibrium in the study of networks, e.g., Myerson (1991), Dutta and Mutuswami (1997), and Bala and Goyal (2000). Our paper falls in the former strand of literature on networks. As it is well argued in Jackson and Nouweland (2005), although the non-cooperative solution concepts such as Nash-based equilibrium solutions may offer powerful predictions and sufficient characterization of economic and social networks, there are a number of reasons to study cooperative notions in this context. Notably, there are many social and economic environments where coalitional interactions among agents in networks are more appropriate. Economic and social network examples of this kind of environment are abundant: formation of international cartels, formation of trade blocs, membership in social clubs, and internet chat rooms and social networks such as Facebook, to mention a few.

We investigate the stability of networks by introducing a notion of *stability* based on von Neumann and Morgenstern (1947) (vN&M, hereafter) solution but somewhat stronger. Although Dutta and Mutuswami (1997) and Jackson and Nouweland (2005) also address the issue of network stability, the notion of *stability* we consider is different. Dutta and Mutuswami (1997) defined a network, given an allocation, to be stable if no coalition of players wants to deviate by forming or severing links. Accordingly, deviations are valid if all members are strictly better off. Jackson and Nouweland (2005) considered a stronger notion of stability under which a deviation is valid if some member of the deviating coalition will be strictly better off while others are weakly better off. The notions of stability that were introduced in both Dutta and Mutuswami (1997) and Jackson and Nouweland (2005) are core-type stability notions. The problem that we raise here is in the spirit of the vN&M's criticism of the notion of core in cooperative game context. That is, we raise an issue with an allocation-network pair to which a coalition may deviate. Let an allocation-network pair be disqualified because it is dominated by another allocation-network pair. In spirit of vN&M (1947), what if this latter allocation-network pair is not stable itself? We should not give an immunity status to this allocation-network pair. Thus, we follow vN&M (1947) and offer a new notion of network stability.

Although we impose an additional requirement for stability of network, it should be noted our notion of network stability is more robust than the core-like stability notions in current literature such as Dutta and Mutuswami (1997). According to these core-type network stability concepts, it is possible that a network would be deemed unacceptable (unstable) if there exists another network that a group of players can induce (by forming or severing links) and be better off by doing so even if the newly formed network is unacceptable (unstable) itself. In a sense, players are myopic as they are unable to see the network they use as an objection to the current network lacks credibility. On the other hand, according to our vN&M-based notion of network stability, a network is unacceptable (unstable) if a group of players can block it by forming another network which itself is stable. Thus, in a sense, an objection to a network is credible in our definition. Moreover, according to our notion of domination a coalition of players can object to a pair of allocation-network if they can induce a new pair under which no player in the coalition is worse off while some members are better off. In contrast, under the original vN&M notion of dominance relation all members of the coalition should be better off. This feature of domination relation is similar to the dominance relation used in Jackson and Nouweland (2005) although they applied it to defines a core-type stability notion. In other words, our notion of stability is stronger than both Jackson and Neuweland (2005) as our notion has a dual stability requirements (i.e., internal and external stability) as in vN&M (1947).

To motivate our vN&M-based notion of network stability, we appeal to a real world example. One of the recent applications of network economics is the formation of free trade areas. If free trade exists among a group of countries, these countries will be connected through links. As an example assume there are four countries in the world: 1, 2, 3, and 4. Consider three networks (i.e., configuration of the world): A, B, and C. Assume we have bilateral free trade between 1 and 2, 1 and 4, 2 and 3, and 3 and 4 under network A. In network B countries 1 and 2 as well as 3 and 4 have bilateral free trade, while in network C we have global free trade whereby any country has bilateral free trade with all other countries. These networks are depicted in Figure 1. For the sake of argument, suppose trading arrangement A has been proposed. That is, let A be the status quo. Assume that countries 1 and 2 can be better off if they terminate their free trade (sever their links) with 4 and 3, respectively. That is, they can induce network B and as a result be better off. With core-type notions of network stability, such an objection would make network A unstable. However, we question: what if the free trade configuration B can be objected via the configuration C by a (weak) subset of countries? In this case it seems unreasonable to give immunity to the configuration B. That is, the objection jointly made by countries 1 and 2 via proposing the configuration B is in a sense not credible. In contrast, based on the concept of network stability that we introduce here, a network will be ruled out as being unstable if some players can object to it via a stable network.

<u>Insert Figure 1 here</u>

In this context, we study the architecture of stable networks by showing the conditions under which *complete* and *star* networks are stable. Interestingly, we will indicate that an egalitarian allocation under a complete network can emerge as stable. While our notion of network stability is compelling, it is equally important to show the conditions under which such stable sets of networks exists. We will also study the conditions for existence as well as uniqueness of stable networks.

Following this introduction, we will present our framework and define economic networks. In Section 3 we will introduce our notion of stability. Section 4 investigates the structure of stable networks. We will address the issues of existence of a stable set of networks in Section 5. We conclude the paper in the last section. All proofs are presented in the appendix.

2 Economic networks

Let the finite set of players be $N = \{1, 2, 3, ..., n\}$. Denote by G^N the set of all subsets of N with size 2. Similarly, for any $S \subset N$, denote by G^S the set of all subsets of S with size 2. The set of all possible networks on N is defined as $G = \{g | g \subset G^N\}$. A link between player $i, j \in N$, denoted by ij, in network g is an element of g. This will be represented by the empty set if there is no link between any two players. For any network $g \in G$ and any $S \subset N$, define a subnetwork $g(S)$ as $g(S) = g \cap G^S$. We say that a player $i \in N$ is an *isolated* player under a network $g \in G$ if $ij \notin g, j \in N$. That is, a player is isolated under a network if he does not have any link with any other player in the network.

A path in network $g \in G$ between players i and j, denoted by $\pi_g(ij)$, is a sequence of linked players $i_1, i_2, ..., i_K$ such that $i_k i_{k+1} \in g, \forall k \in \{1, 2, ..., K-1\}$ where $i_1 = i$ and $i_K = j$. The set of all paths between players i and j in network g is denoted by $\Pi_g(ij)$. A component of a network $g \in G$ is a subnetwork $g(S) \subset g, S \subset N$, where $i, j \in S$ if and only if there exists a path $\pi_g(ij) \in \Pi_g(ij)$. Denote the set of all components of any network $g \in G$ by $C(g)$.

The value of a network is defined by a value function $\nu : G \mapsto \mathbb{R}_+$. We normalize the value function such that $\nu(\emptyset) = 0$. For any $g \in G$ and any permutation of the set of players $\pi : N \mapsto N$, define $g^\pi = \{ij | i = \pi(k), j = \pi(l), kl \in g\}$.[1] We say that a value function ν is *anonymous* if $\nu(g) = \nu(g^\pi)$. We say that a value function ν is *convex* if for all g_1, g_2, and $g_3 \in G$, if $g_1 \subset g_2 \subset g_3$, then $\nu(g_3) - \nu(g_2) \geq \nu(g_2) - \nu(g_1)$. On the other hand, a value function $\nu(g)$ is *non-convex* if there exist g_1, g_2, and $g_3 \in G$ such that $g_1 \subset g_2 \subset g_3$, we have $\nu(g_3) - \nu(g_2) < \nu(g_2) - \nu(g_1)$.

Denote the set of all allocations given the network $g \in G$ by $A|g$, i.e., $A|g = \{x \in \mathbb{R}^n_+ | \sum_{i \in N} x_i \leq$

[1] That is, π is a bijection from N to N.

$\nu(g)\}$. We also denote for any $g \in G$ and proposed allocation $x \in A|g$ by $x|g$. We simply refer to this as an allocation-network pair. Let Ω be the set of all allocation-network pairs, i.e., $\Omega = \{x|g \quad |g \in G, x \in A|g\}$. Consider networks $g, g' \in G$. We say that g' is reachable from g via $S \subset N$, denoted by $g \longmapsto_S g'$ if:

1) $ij \in g' \setminus g$, then $i, j \in S$; and

2) $ij \in g \setminus g'$, then i and/or $j \in S$.

3 Stability

One of the most important and widely used solution concepts in cooperative game theory is the notion of core. We first define this notion in the context of economic networks.

Definition 1. *A network $g \in G$ is a core network with respect to allocation $x \in A|g$ if there do not exist $g' \in G$, $S \subset N$, $g \longmapsto_S g'$, $g'(S) \in C(g')$ and $y \in A|g'$ such that $\sum_{i \in S} y_i \leq \nu(g'(S))$ and $y_i > x_i \forall i \in S$.*

Denote the set of all core allocation-network pairs by $\mathbb{C}$. According to the concept of core networks, an allocation-network pair $x|g$ is disqualified to be in the set of core allocation-network pairs if there exists another allocation-network pair $y|g'$ and a subset of players such that these players can induce g' from g and all be better off under $y|g'$. The problem we raised in the introduction is that what if $y|g'$ is itself dominated by yet another allocation-network pair? In such a case, $y|g'$ should not be treated with immunity. vN&M (1947) addresses this criticism by introducing a notion of stability in the context of cooperative game theory. Similarly, we define a notion of stability in the context economic networks.

As stated in the introduction, although we propose a vN&M-based notion of stability in the context of network, our stability concept is stronger than what vN&M originally introduced as we use a stronger dominance relation to define stability. In contrast to the dominance relation in vN&M stability notion, if a coalition of players can induce a pair of allocation/network by which some members are better off while other members are not worse off, there is no reason for the latter group of members not to go along. Thus, we re-define dominance relation in the context of networks as follows. We say that a network $g' \in G$ dominates another network $g \in G$ given $x \in A|g$ if there exist $S \subset N$, $g \longmapsto_S g'$, $g'(S) \in C(g')$ and $y \in A|g'$ such that $\sum_{i \in S} y_i \leq \nu(g'(S))$, $y_i \geq x_i \forall i \in S$, and $y_i > x_i$ for some $i \in S$. Denote by $\Delta(x|g)$ the set of allocation-network pairs that dominate allocation-network pair $x|g$. Clearly, it follows from the definition of core allocation-network pair, an allocation-network $x|g \in \mathbb{C}$ if $\Delta(x|g) = \emptyset$. It should be note that that the converse of this statement is not generally true. The following definition formally introduces our notion of network stability.

Definition 2. *A set of allocation-network pairs $V \subset \Omega$ is internally stable if for any $x|g \in V$ there does not exist $y|g' \in V$ such that $y|g' \in \Delta(x|g)$. A set of allocation-network pairs $V \subset \Omega$ is externally stable if for any $z|g' \in \Omega \setminus V$, and $V \neq \emptyset$, there exists an allocation-network pair $x|g \in V$ such that $x|g \in \Delta(z|g')$. A set of allocation-network pairs V is stable if it is both internally and externally stable.*

Internal stability addresses internal consistency. In other words, among the set of internally stable allocations-network pairs, no subset of players can induce another (stable) allocation-network pair that some members of the subset prefer while other members are at least indifferent. On the other hand, external stability addresses external consistency. That is, there should be a reason a pair of allocation-network is deemed unacceptable by a subset of players. Such players can induce another pair of allocation-network which some such players prefer while other members of the subset are at least indifferent and this allocation-network pair is itself acceptable (i.e., stable).[2]

Example 1:

Assume $N = \{1, 2, 3\}$. Define value function ν as:

$\nu(\{12, 13, 23\}) = 3$

$\nu(\{12, 13\}) = \nu(\{12, 23\}) = \nu(\{13, 23\}) = 1$

$\nu(\{ij\}) = 0.5, i, j \in N, i \neq j.$

Let $\tilde{g} = \{12, 13, 23\}$ and define $V = \{(x|\tilde{g})|\Sigma_{i\in N} x_i = 3\}$. Set of allocation-network pairs V is stable. Since set V contains only one network, under which all its Pareto allocations included, the internal stability is trivial. To verify the external stability, note that for a network with a structure $g^2 = \{ij, jk\}$ any player can at most get an allocation of 1, and the remaining players can get a total of what is left of 1. Denote such an allocation by y where $y_i \geq 0$ and $\Sigma_{i\in N} y_i = 1$. Now consider allocation $\tilde{x}$ such that $\tilde{x}_i = y_i + 2/3$. Clearly, $\Sigma_{i\in N}\tilde{x}_i = \nu(\tilde{g})$ and $\tilde{x}_i > y_i, \forall i \in N$. Moreover,

[2] As it can be seen, our notion of stability is stronger than that of Jackson and Nouweland (2005) defined in the context of networks due to our dual stability requirements (internal and external).

$\tilde{x}_i|\tilde{g} \in V$ and $\tilde{x}_i|\tilde{g} \in \Delta(y|g^2)$. A similar argument can be made for all networks with structure $\{ij\}$ and their feasible allocations. Therefore, V is externally stable.

Although it is well-known that a core allocation of a cooperative game (if nonempty) is in its vN&M solution, it is also interesting to relate the set of core allocation-network pairs with stable set of allocation-network pairs. The following result addresses this relationship.

Claim 1. *Let V be any set of stable allocation-network pairs. Then, $\mathbb{C} \subset V$.*

4 Network structure

Having defined the stability of economic networks, we now turn to network architecture. Under what condition are various network structures and allocations stable? In this section, more specifically, we will study the conditions under which a complete network and a star network along with a set of allocations are stable. We say that a network is complete, denoted by g_c, if for any pair of players $i, j \in N$, $ij \in g_c$. That is, a network is complete if there is a link between every pair of players. We say that a network is a star network, denoted by g_s, if it does not have an isolated player and there exists a player $i \in N$ such that for every link $jk \in g_s$ we have $j = i$. Player i is called the central player, and all player $k \in N \setminus \{i\}$ are peripheral players.[3] We say that a feasible allocation is egalitarian, denoted by x^e, under a network $g \in G$ if $x_i^e = \nu(g)/|N|$.

The following result characterizes the set of core allocation-network pairs. It highlights that a condition under which core networks are complete networks.

Claim 2. *Let value function ν be convex and $\mathbb{C}$ be non-empty. Then, $\forall x|g \in \mathbb{C}$ network g is a complete network.*

The following lemma will be useful in characterizing architecture of stable networks.

Lemma 1. *Let a value function ν be convex. Then, $\nu(g)/|N| \geq \nu(g(S))/|S|$, $\forall S \subset N$ and $\forall g \in G$ if there is no isolated player under g.*

This lemma addresses an important property of a convex value function. It states that the per-member value of any network g is no less that the per-member value of any its nontrivial

[3] The central player in a network is a reminiscent of Kalai, Polstlewaite, and Roberts notion of the middleman (see Kalai et al. (1978)).

subnetworks (i.e., a $g(S) \neq g$, $S \subset N$) given that the value function is convex and the network does not have an isolated player. As stated earlier, this lemma plays a crucial role in the architecture of a stable network.

First, consider complete networks. This type of network structure is perhaps the most appealing form of economic or social network architectures, as it is the most inclusive. The ideal of inclusiveness is a major characteristic of some important international clubs (or organizations) such as United Nations and, to a lesser extend, the World Trade Organization. Therefore, it is of paramount importance to study the conditions for stability of this type of networks. The following theorem addresses this issue.

Theorem 1. *Assume that the value function ν is convex and let $V \subset \Omega$ be a set of stable allocation-network pairs. Then, the complete network with egalitarian allocation is stable, i.e., $x^e|g_c \in V$.*

According to the above result, the complete network is among the set of stable networks if the value function is convex. In addition, this condition guarantees that egalitarian allocation is also among the set of allocations supported by the stable complete network. The notion of equity has been a focal issue, and perhaps controversial, since the dawn of human civilization. The above theorem reconsiders such a notion in the context of a network from the stability point of view.

Example 2:

In this example we show that the assumption of convexity is essential for validity of the results of Theorem 1. Assume $N = \{1, 2, 3, 4\}$. Define the value function ν as:

$\nu(\{12, 23\}) = 3$

$\nu(\{13, 24\}) = 2.4$

$\nu(\{13\}) = \nu(\{24\}) = 1.2$

$\nu(g) = 0$ for all other networks g.

Consider $\tilde{g} = \{12, 23\}$. We claim that $V = \{x|\tilde{g} \quad |x_1 + x_2 + x_3 = 3, x_4 = 0\}$ is a stable set of allocation-network pairs. Internal stability is trivial. To show the validity of external stability of V, it is enough to consider the following cases:

i) Consider network $g^1 = \{13, 24\}$. For any $y \in A|g^1$, we have $\Sigma_{i \in N} y_i = 2.4$. Now, consider the subset of players $S = \{1, 2, 3\}$ and construct allocation $\tilde{x}$ such that $\tilde{x}_i = y_i + (0.6 + y_4)/3$ for all $i \in S$ and $\tilde{x}_4 = 0$. Clearly, $\tilde{x} \in A|\tilde{g}$. Moreover, players 2 and 3 can sever their links with 4 and

1, respectively, and form new link with each other as well as 2 with 1. Clearly, $\tilde{g}(S) \in C(\tilde{g})$. In addition, it is evident by construction that $\Sigma_{i \in S} \tilde{x}_i = 3$, and $\tilde{x}_i > y_i, \forall i \in S$. Thus, $\tilde{x}|\tilde{g} \in \Delta(y|g^1) \cap V$.

ii) Consider network $g^2 = \{13\}$ ($g^3 = \{24\}$). We have $\Sigma_{i \in N} y_i = 1.2, \forall y \in A|g^2 (\forall y \in A|g^3)$. Now construct allocation $\tilde{x}$ such that $\tilde{x}_i = y_i + (1.8 + y_4), \forall i \in S$ and $\tilde{x}_4 = 0$. Players 1 and 3 can sever their link and each can form a new link with player 2 (Player 2 can sever her link with 4 and form new links with 1 and 3). Thus, subset of players $S = \{1, 2, 3\}$ can induce $\tilde{g}$ as $\tilde{g}(S) \in C(\tilde{g})$. Moreover, by construction, we have $\Sigma_{i \in S} \tilde{x}_i = 3$ and $\tilde{x}_i > y_i, \forall i \in S$. It follows that $x|\tilde{g} \in \Delta(y|g^1) \cap V$.

Next, we consider star networks, another type of network structure with numerous interesting applications in economics and social settings. Hub-and-spoke - where a central member (the hub) is connected to all other peripheral members (the spokes) - is a classic example of this kind of network. As interesting economic example is the numerous bilateral preferential trading agreements formed during the past couple of decades between the United States, the hub, and other countries, the spokes, which did not have any form of preferential trade agreement among themselves (for instance, Israel and Jordan). The following theorem considers the stability condition for star networks.

Theorem 2. *Assume an anonymous value function ν. A star network g_s given allocation x is a stable network-allocation pair (i.e., $x|g_s \in V$, for any stable set of allocation-network pairs $V \subset \Omega$) only if value function ν is non-convex.*

5 Existence

In this section we shall address the existence problem for sets of stable allocations-network pairs. As in Jackson and Nouweland (2005), we relate the notion of network with an associated cooperative game. By Doing so, we shall take advantage of substantial development in cooperative game theory.

A pair (N, ω) is a cooperative game where $N = \{1, 2, 3, ..., n\}$ is the set of players and $\omega : 2^N \mapsto \mathbb{R}$ is a characteristic function. An allocation $x \in \mathbb{R}^n$ is feasible if $\sum_{i \in N} x_i \leq \omega(N)$. A feasible allocation x is a core allocation for (N, ω) if $\Sigma_{i \in S} x_i \geq \omega(S)$ for all $S \subset N$. Denote the set of core allocations for game (N, ω) by $\mathbb{C}_\omega$. A set of feasible allocations X is internally stable if for all $x \in X$ there does not exist $y \in X$, $S \subset N$ such that $\sum_{i \in S} y_i \leq \omega(S)$ and $y_i > x_i \forall i \in S$. A set of feasible allocations X is externally stable if for all feasible allocations $y \notin X$ there exist a feasible allocation $x \in X$ and $S \subset N$ such that $\sum_{i \in S} x_i \leq \omega(S)$ and $x_i > y_i \forall i \in S$. A Set of feasible allocation X is stable if it is both internally and externally stable. Finally, for any value function ν and set of players N define an associated cooperative game (N, Ω^{ν}) where $\omega^{\nu} = \max_{g \in G^S} \nu(g)$ for all $S \subset N$. The following theorem relates the set of core allocation-network pairs with the set of core allocations of an associated cooperative game.

Theorem 3. *An allocation $x \in \mathbb{R}^n_+$ is in core of (N, ω^{ν}) if and only if there exists $g \in G$ where $x|g \in \mathbb{C}$.*

We now turn to conditions under which a stable set of allocation-network pairs exists. It turns out that convexity of a value function guarantee that a stable set of allocation-network pair exists. Moreover, and equally importantly, such a set is unique and non-empty. The following theorem formally addresses these results.

Theorem 4. *Let value function ν be convex. Then, $\mathbb{C} \neq \emptyset$ is a unique stable set of allocation-network pairs.*

This result states that if a value function is convex, then the set of core allocation-network pairs is a stable set of allocation-network. In addition, not only does this set is a unique stable set of allocation-network pairs, it is also non-empty.

6 Conclusion

This paper introduced a notion of stability of economic networks based on von Neumann and Morgenstern (1947) solution although the notion of domination we used to define our notion of network stability differs from von Neumann and Morgenstern (1947) and resembles that of Jackson and Nouweland (2005). Apart from this departure from von Neumann and Morgenstern (1947), our notion network stability also demands stronger stability requirements than those in the current literature on economic networks (e.g., Dutta and Mutuswami (1997) and Jackson and Nouweland (2005)), and is more appealing as it does not suffer from core-type criticism. This latter feature of our stability notion is in the spirit of von Neumann and Morgenstern's (1947) criticism of core.

As our main results, we studied the architecture of stable networks. More specifically, we analyzed the conditions under which complete or star networks are stable. Assuming a convex value function, we showed that the egalitarian allocation under a complete network is stable. Moreover, we addressed the existence of a stable set of allocations-network pairs. As is the case for vN&M solution, existence and uniqueness are not guaranteed. Usually, restrictive assumptions are to imposed for existence of a vN&M solution of a cooperative game. We showed that if a value function is convex, then there exists a unique and non-empty stable set allocation-network pairs. More importantly, we show that such a unique set is in fact the set of core allocation-network pairs.

The framework presented in this paper can be extended in several directions. First, one can extend our ideas to one-sided networks. Second, our notion of stability can be applied to issue of economic integration by analyzing the conditions under which various forms of trade blocs can be stable following the rationale we presented in this paper. Third, our notion of stability may be applied in the arena of international relations such as formation of international security arrangements, among others.

Appendix: Proofs

Proof of Claim 1. Let V be any set of stable allocations-network pairs. Assume in negation that there exists a core allocation-network pair $x|g \in C \setminus V$. Then, by external stability of V, there must exist an allocation-network pair $y|g' \in V$ such that $y|g' \in \Delta(x|g)$. This, in turn, implies that there exist a subset of players $S \subset N$, $g' \in G$, and $y \in A|g'$ such that $g \longmapsto_S g'$, $g'(S) \in C(g')$, $\sum_{i \in S} y_i \leq \nu(g'(S))$, $y_i \geq x_i \forall i \in S$, and $y_i > x_i$ for some $i \in S$. Without loss of generality assume that for a single player $k \in S$ we have $y_k > x_k$ and $y_i = x_i \forall i \in S \setminus \{k\}$. Now construct allocation $z \in A|g'$ such that $z_i = y_i \ \forall i \in N \setminus S$, $z_k = y_k - \epsilon$ and for all $i \in S \setminus \{k\}$ let $z_i = y_i + \epsilon/(|S| - 1)$, for sufficiently small positive $\epsilon < y_k - x_k$. Clearly, by construction we have $\sum_{i \in S} z_i = \sum_{i \in S} y_i$ and $z_i > x_i \forall \in S$. Thus, it follows that $\sum_{i \in S} z_i \leq \nu(g'(S))$. All these conclude that there exist $S \subset N$ and an allocation-network pair $z|g'$ such that $g \longmapsto_S g'$, $g'(S) \in C(g')$, $\sum_{i \in S} z_i \leq \nu(g'(S))$, and $z_i > x_i \forall i \in S$. This contradicts our negation assumption that $x|g \in \mathbb{C}$. $\square$

Proof of Claim 2. Assume the negation. That is, g is not complete for some $x|g \in \mathbb{C}$. Now, consider complete network g_c and construct allocation $y \in \mathbb{R}^n_+$ such that $y_i = x_i + [\nu(g_c) - \nu(g)]/|N|$. By convexity of ν we have $\nu(g_c) > \nu(g)$ because $g \subset g_c$, implying that $y_i > x_i, \forall i \in N$. Also, by construction we have $\sum_{i \in N} y_i = \nu(g_c)$. Finally, it is direct that $g \longmapsto_N g'$ and $g_c(N) \in C(g_c)$. All these imply that $x|g \in \Omega \setminus \mathbb{C}$, contradicting with $x|g \in \mathbb{C}$. $\square$

Proof of Lemma 1. Let $N \setminus S = \{k_1, k_2, k_3, ..., k_J\}$. Construct network $g_{k1} \equiv g(S) \cup \{ik_1\}$, where $ik_1 \in g \setminus g(S)$, for some $i \in N$. Note that as there is no isolated player under g, such a link exists. By convexity $\nu(g_{k1}) - \nu(g(S)) \geq \nu(g(S)) - \nu(\emptyset)$, indicating that $\nu(g_{k1})/|S \cup \{k_1\}| \geq 2\nu(g(S))/|S \cup \{k1\}| \geq \nu(g(S))/(|S|)$, where the last inequality is due to the fact that $2/|S \cup \{k1\}| = 2/(|S|+1) \geq 1/|S| \Leftrightarrow 2|S| \geq |S| + 1$ since $|S| \geq 1$. Thus, $\nu(g_{k1})/|S \cup k_1| \geq \nu(g(S))/|S|$. Similarly, construct $g_{k2} \equiv g_{k1} \cup \{ik_2\}$, where $ik_2 \in g \setminus g(S)$ for some $i \in N$. Again by convexity, we have $\nu(g_{k2}) - \nu(g_{k1}) \geq \nu(g_{k1}) - \nu(g(S))$. This implies that $\nu(g_{k2}) \geq 2\nu(g_{k1}) - \nu(g(S)) \geq \nu(g_{k1}) + \nu(g(S)) \geq 3\nu(g(S))$, where the last two inequalities are due to the fact that $\nu(g_{k1}) \geq 2\nu(g(S))$. It then follows that $\nu(g_{k2})/|S \cup \{k_1, k2\}| \geq 3\nu(g(S)))/|S \cup \{k_1, k_2\}|$. On the other hand, $3\nu(g(S)))/|S \cup \{k_1, k_2\}| \geq \nu(g(S)))/|S|$ because $3/|S \cup \{k_1, k_2\}| = 3/(|S| + 2) \geq 1/|S| \Leftrightarrow 3|S| \geq |S| + 2$ since $|S| \geq 1$. All these imply that $\nu(g_{k2})/(|S \cup \{k_1, k_2\}|) \geq \nu(g(S))/|S|$. Now assume that $\nu(g_{kJ-1})/|N \setminus \{k_J\}| \geq \nu(g(S))/|S|$

where g_{kJ-1} is defined as g_{k1}, g_{k2}, etc. as above. Similar to the argument we just made for g_{k1} and g_{k2}, it can be shown that $\nu(g_{kJ})/|N| \geq \nu(g(S))/|S|$ where $g_{kJ} \equiv g_{kJ-1} \cup \{ik_J\}$ and $ik_J \in g \setminus g(S)$. Therefore, since N is a finite set and by induction, for $\forall l \in N$ and $g_{lm} \subset g$ (as constructed above), $lm \in g \setminus g(S)$, and $m \in N \setminus S$, we have $\nu(g_{lm})/|S \cup \{m\}| \geq \nu(g(S))/|S|$. This concludes the lemma. □

Proof of Theorem 1. In light of Claim 1 it is enough to show that $x^e|g_c \in \mathbb{C}$. To prove this, assume the negation; i.e., $x^e|g_c \in \Omega \setminus \mathbb{C}$. That is, there exist $S \subset N$ and $g' \in G$ such that $g_c \longmapsto_S g'$, $g'(S) \in C(g')$, $y \in A|g'$, $\sum_{i \in S} y_i \leq \nu(g(S))$ and $y_i > x_i^e, \forall i \in S$. As g_c is a complete network, it follows that for some $i, j \in S$, either $ij \notin g'$ and/or $ik \notin g'$ for some $k \in N \setminus S$. That is, members of S can induce g' by only severing links among themselves and/or links with other players. It also follows from our negation assumption that for all $i \in S$, $y_i > x_i^e = \nu(g_c)/|N|$. Let $\bar{y} = \min\{y_i | i \in S\}$. By the definition of domination, we conclude that $\bar{y} \leq \nu(g'(S))/|S|$ since $\sum_{i \in S} y_i \leq \nu(g'(S))$. As $g' \subset g_c$, it follows from convexity of ν that $\nu(g_c(S)) \geq \nu(g'(S))$. Then, it follows that $\nu(g_c(S))/|S| \geq \nu(g'(S))/|S| \geq \bar{y} > x_i^e = \nu(g_c)/|N|$. That is, $\nu(g_c(S))/|S| > \nu(g_c)/|N|$, contradicting the convexity of ν due to Lemma 1 since there is no isolated player under a complete network. □

Proof of Theorem 2. Assume the negation. That is, assume an anonymous convex value function ν and let V be a stable set of allocation-network pair where $g_s|x \in V$ for some $x \in A|g_s$. Without loss of generality let player 1 be the central player. Construct the following sequence of allocation-network pairs.

- $g^1 = g_s$, $y^1 = x$
- $g^2 = g^1 \bigcup \{23\}$, $y^2 \in A|g^2$ such that $y_j^2 = y_j^1 + \frac{\nu(g^2) - \nu(g^1)}{2} \quad \forall j \in \{2,3\}$ and $y_j^2 = y_j^1 \quad \forall j \in N \setminus \{2,3\}$
- $g^3 = g^2 \bigcup \{24\}$, $y^3 \in A|g^2$ such that $y_j^3 = y_j^2 + \frac{\nu(g^3) - \nu(g^2)}{2} \quad \forall j \in \{2,4\}$ and $y_j^3 = y_j^2 \quad \forall j \in N \setminus \{2,4\}$
- ...

- $g^{n-1} = g^{n-2} \bigcup \{2n\}$, $y^{n-1} \in A|g^{n-1}$ such that $y_j^{n-1} = y_j^{n-2} + \frac{\nu(g^{n-1})-\nu(g^{n-2})}{2} \quad \forall j \in \{2, n\}$ and $y_j^{n-1} = y_j^{n-2} \quad \forall j \in N \setminus \{2, n\}$

- ...

- $g^k = g^{k-1} \bigcup \{i(i+1)\}$, $y^k \in A|g^k$ such that $y_j^k = y_j^{k-1} + \frac{\nu(g^k)-\nu(g^{k-1})}{2} \quad \forall j \in \{i, i+1\}$ and $y_j^k = y_j^{k-1} \quad \forall j \in N \setminus \{i, i+1\}$

- ...

- $g^M = g^{M-1} \bigcup \{(n-1)n\}$, $y^M \in A|g^M$ such that $y_j^M = y_j^{M-1} + \frac{\nu(g^M)-\nu(g^{M-1})}{2} \quad \forall j \in \{(n-1), n\}$ and $y_j^M = y_j^{M-1} \quad \forall j \in N \setminus \{(n-1), n\}$

where $M = [n!/(n-2)!]/2 - (n-2)$ is finite since n is finite. Note also that $g^M = g_c$. Due to convexity, we then have $\Sigma_{i \in \{2,3\}} y_i^2 \geq \nu(\{23\})$ because $\nu(g^2) - \nu(g^1) \geq \nu(g^1) \geq \nu(g^1(\{1i\}) \geq \nu(\{1i\}) = \nu(\{23\})$. The inequalities are followed from convexity and the equality is due to anonymity. Similarly, we conclude that $\Sigma_{i \in \{2,4\}} y_i^3 \geq \nu(\{24\})$ because $\nu(g^3) - \nu(g^2) \geq \nu(g^2) \geq \nu(g^1) \geq \nu(g^1(\{1i\})) \geq \nu(\{1i\}) = \nu(\{24\}) \quad \forall i \in N \setminus \{1\}$. Again, the last equality is due to anonymity. Moreover, $\Sigma_{i \in \{2,3,4\}} y_i^3 \geq \nu(\{23, 24\})$ because $\nu(g^3) - \nu(g^1) \geq \nu(g^1) \geq \nu(g^1) \geq \nu(g^1(\{1i, 1j\})) \geq \nu(\{23, 24\}) \quad \forall i, j \in N \setminus \{1\}$. In the same fashion we conclude that $\Sigma_{i \in \{2,n\}} y_i^n \geq \nu(\{2n\})$ because $\nu(g^n) - \nu(g^{n-1}) \geq \nu(g^{n-1}) \geq \nu(g^1) \geq \nu(g^1(\{1i\})) \geq \nu(\{1i\})) = \nu(\{2n\}) \quad \forall i \in N \setminus \{1\}$. In addition, $\Sigma_{i \in N \setminus \{1\}} y_i^n \geq \nu(\{2i | i \in N \setminus \{1\}\})$, $\Sigma_{i \in N \setminus \{1,n\}} y_i^n \geq \nu(\{2i | i \in N \setminus \{1, n\}\})$, ..., $\Sigma_{i \in \{2,3,4\}} y_i^n \geq \nu(\{23, 24\})$. By continuing this line of argument we have $\Sigma_{i \in \{n, n-1\}} y_i^M \geq \nu(\{n, n-1\})$ because $\nu(g^M) - \nu(g^{M-1}) \geq \nu(g^{M-1}) \geq \nu(g^1) \geq \nu(g^1(\{1i\})) \geq \nu(\{1i\}) = \nu(\{n(n-1)\}) \quad \forall i \in N \setminus \{1\}$ and $\Sigma_{i \in N \setminus \{1\}} y_i^M \geq \nu(\{ni | i \in N \setminus \{1, n\})$, $\Sigma_{i \in N \setminus \{1,n\}} y_i^M \geq \nu(\{(n-1)i | i \in N \setminus \{1, n-1\}\})$, ..., $\Sigma_{i \in \{2,3,4\}} y_i^M \geq \nu(\{23, 24\})$. That is, by construction, we have $\Sigma_{i \in S} y_i^M \geq \nu(g(S)), \forall g \in G, \forall S \subset N$. This implies by external stability of V that $y^M | g^M \in V$. It also follows from our construction that $y_i^M > y_i^1, \forall i \in N \setminus 1$ and $y_1^M = y_1^1$. Moreover, by the definition of a complete network we have $g_c = g_c(N) \in C(g_c)$, i.e., set N can induce g_c from g_s. Thus, $y^M | g^M \in \Delta(x | g_s) \cap V$, which contradicts internal stability of V. $\square$

Proof of Theorem 3. First, Let $x \in \mathbb{R}^n$ be a core allocation for game (N, ω^ν). In negation, let $x | g \in \Omega \backslash \mathbb{C}, \forall g \in G$. That is, there does not exist $g \in G$ such that $x | g \in \mathbb{C}$. Our negation assumption implies that $\exists S \subset N, g' \in G, y \in A|g'$ such that $g \longmapsto g', g'(S) \in C(g), \sum_{i \in S} y_i \leq \nu(g'(S))$ and $y_i > x_i, \forall i \in S$. However, since $\omega^\nu(S) = \max_{g \in G^S} \nu(g)$, it must be the case that $\omega^\nu(S) \geq \nu(g'(S))$. Now fix allocation x such that $z_i = y_i + [\omega(S) - \nu(g'(S))]/|S|$ for all $i \in S$ and $z_i = y_i, \forall i \in N \setminus S$. Clearly, by construction, we have $\sum_{i \in S} z_i \leq \omega^\nu(S)$ and $z_i \geq y_i > x_i, \forall i \in S$. This contradicts with x being a core allocation for (N, ω^ν).

Second, assume that $x|g \in \mathbb{C}$ but in negation let allocation x not be a core allocation for game (N, ω^ν). By negation assumption and the definition of ω^ν, there must exist a network $g' \in G$ and $S \subset N$ such that $g'(S) \in \arg\max_{g \in G^S} \nu(g)$, thus $g'(S) \in C(g')$, and exist an allocation y such that $\sum_{i \in S} y_i \leq \omega^\nu(S) = \nu(g'(S))$ and $y_i > x_i, \forall i \in S$. Since $\sum_{i \in S} y_i \leq \nu(g'(S)), g'(S) \in C(g')$ and $y_i > x_i, \forall i \in S$, we have $x|g \notin \mathbb{C}$ which is a contradiction. □

Proof of Theorem 4. First, we show that $\mathbb{C}_\omega$ is a unique vN&M solution of game (N, ω^ν) if ν is convex. Since it is well-known that the set of core allocations of a cooperative game is its unique vN&M solution if its characteristic function is convex, it is sufficient to show that if value function ν is convex then ω^ν is convex.[4] To prove this, let ν be convex and fix any arbitrary $S \subset T \subset N$. By convexity of ν, for any $g \in G$ and $i \in S \subset T$ we have $\nu(g(T)) \geq 2\nu(g(T \setminus \{i\}))$ and $\nu(g(T)) \geq 2\nu(g(S))$. These inequalities imply that $\nu(g(T)) \geq \nu(g(T \setminus \{i\})) + \nu(g(S))$. Since $\nu(g(S \setminus \{i\} \geq 0$, we conclude that $\nu(g(T)) \geq \nu(g(T \setminus \{i\})) + \nu(g(S)) - \nu(g(S \setminus \{i\}) \iff \nu(g(T)) - \nu(g(T \setminus \{i\})) \geq \nu(g(S)) - \nu(g(S \setminus \{i\})$. Since this inequality is true for any artbitrary $g \in G$ and $i \in S \subset T \subset N$, it implies that $\omega^\nu(T) - \omega^\nu(T \setminus \{i\}) \geq \omega^\nu(S) - \omega^\nu(S \setminus \{i\})$. That is, ω^ν is convex.

Next, we show that $\mathbb{C}$ is stable. Since $x|g \in \mathbb{C}$ if $\Delta(x|g) = \emptyset$, internal stability of $\mathbb{C}$ is direct. Therefore, it is enough to show that $\mathbb{C}$ is externally stable. Let $x|g \in \Omega \setminus \mathbb{C}$. It follows from Theorem 3 that $x \notin \mathbb{C}_\omega$. By external stability of $\mathbb{C}_\omega$ there must exist $S \subset N, y \in \mathbb{C}_\omega$ such that $\sum_{i \in S} y_i \leq \omega^\nu$ and $y_i > x_i, \forall i \in S$. Then, Claim 2 and Theorem 3 imply that $y|g_c \in \mathbb{C}$. If $S = N$, then $y|g_c \in \mathbb{C} \cap \Delta(x|g)$, i.e., $\mathbb{C}$ is externally stable. It is left to show that $S = N$. Assume the negation, i.e., $S \neq N$. Construct allocation $z \in \mathbb{R}^n_+$ such that $z_i = y_i + [\omega^\nu(N) - \omega^\nu(S)]/|N|, \forall i \in N$. Since $\max_{g \in G^N} \nu(g) > \max_{g \in G^S} \nu(g)$ due to convexity of ν, by construction we have $z_i > y_i, \forall i \in N$ implying that $y \notin \mathbb{C}_\omega$, which is a contradiction. Therefore, $S = N$ implying that $\mathbb{C}$ is externally stable.

[4] A characteristic function ω is convex if $\omega(T) - \omega(T \setminus \{i\}) \geq \omega(S) - \omega(S \setminus \{i\})$ for all $i \in S \subset T \subset N$ (see Shapley (1971)).

Finally, we need to show that $\mathbb{C}$ is unique and non-empty. In light of Theorem 1, note that $x^e|g_c \in \mathbb{C}$. Thus, we conclude that $\mathbb{C} \neq \emptyset$. To prove the uniqueness, assume the negation. In particular, assume that there exists another stable set of allocation-network pairs $V \neq \mathbb{C}$. It follows from Claim 1 that $\mathbb{C} \subset V$. Consider an allocation-network pair $x|g \in V \setminus \mathbb{C}$. By external stability of $\mathbb{C}$, there must exist an allocation-network pair $y|g'$ such that $y|g' \in \mathbb{C} \cap \Delta(x|g)$. But, since $\mathbb{C} \subset V$, then $y|g' \in V$. That is, we have $x|g \in V$ and $y|g' \in V$, where $y|g' \in \Delta(x|g)$, contradicting internal stability of V. $\square$

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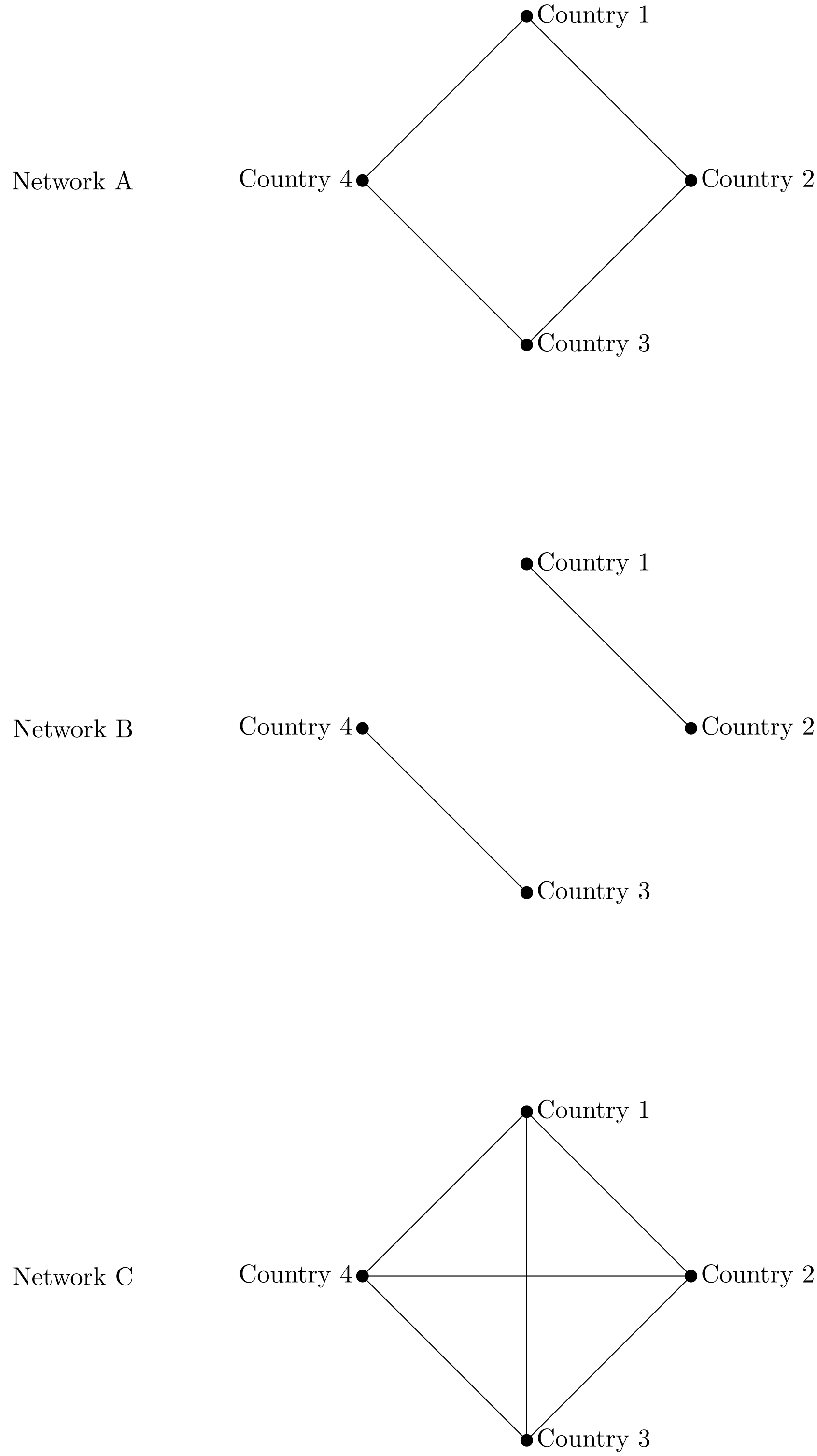

Figure 1: Free Trade Networks