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Tests regarding parameters of several independent gamma populations

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ABSTRACT

Gamma distribution has been used with considerable success in reliability studies and life testing experiments. This distribution assumes various forms as its shape parameter varies which makes it suitable for analyzing a variety of lifetime data. In such studies, often interest lies in comparing a lifetime distribution over multiple groups. In this paper, we develop asymptotic tests for comparing shape parameters of k independent gamma distributions. We also develop similar tests to compare both, the shape and scale parameters, simultaneously. The tests are based on generalized minimum chi-square procedure. This procedure has been known to produce estimators which are asymptotically efficient and the tests based on such estimators are known to have high asymptotic power.

Keywords: Gamma distribution, minimum chi-square, general linear hypothesis, shape and scale parameters, multiple groups.

JEL Classification : C12

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1. Introduction

Gamma distribution has been widely used as a model for reliability and life testing in many areas of applications (Nelson (1982)). It has been used to model length of life of components in household items as well as complex systems such as the rotor blades of helicopters. It arises as a waiting time distribution regarded as a sum of independent exponential inter arrival times in a process with Poisson arrivals. Gamma distribution has been used in industrial engineering and quality control (Drenick (1960), Gupta and Groll (1961)), in cloud seeding experiment (Crow(1977)), and in survival analysis (Gross and Clark (1975), Lawless (2003), Kalbfleisch and Prentice (2002)). The failure rate of gamma distribution is flexible, it is monotone increasing if the shape parameter is greater than 1, and monotone decreasing if the shape parameter is less than 1. Its failure rate is constant when the shape parameter is equal to 1.

The parameters of a gamma distribution are often estimated by the method of maximum likelihood which are obtained as the solutions of nonlinear equations (Bowman and Shenton (1983, 1988), Greenwood and Durand (1960)). Other estimators, such as those based on the method of modified moments and the method of maximum likelihood have also been developed (see for example Cohen and Whitten (1982, 1988). Dahiya and Gurland (1978) developed minimum chi-square estimators which are obtained as solutions of linear equations.

Some procedures are available for testing hypotheses regarding parameters of a gamma distribution. Engelhardt and Bain (1977) developed a test for the scale parameter of a gamma distribution when its shape parameter is unknown. Grice and Bain (1980) presented an asymptotic test regarding the mean while Keating et. al. (1990) developed a test for the shape parameter.

Shiue and Bain (1983) developed an approximate test for comparing the scale parameters of two gamma distributions assuming a common unknown shape parameter. This restriction

was later relaxed in Shiue and Bain (1988). Tripathi et. al. (1993) developed asymptotic procedures based on minimum chi-square for comparing scale parameters as well as coefficients of variation of two or more gamma populations.

Although, there are various tests available for comparing scale parameters of several gamma populations, there are fewer alternatives available for comparing (i) the shape parameters, and (ii) the shape and scale parameters simultaneously. In this paper, we develop asymptotic procedures based on minimum chi-square to compare shape parameters as well as a simultaneous comparison of both the parameters for two or more gamma populations.

The paper is organized as follows. In section two, we develop the minimum chi-square procedure for testing general linear hypothesis. We present the test statistic and its asymptotic distribution. In section three, we specialize the procedure for testing equality of shape parameters, and simultaneously for testing equality of shape and scale parameters of several gamma populations. In section 4, we compute asymptotic power of these tests for a grid of relevant parameter values. In section five, we present an example to illustrate the procedure. Finally, in section six, we present conclusions.

2. Formulation of the test based on Generalized Minimum chi-square procedure

Consider $m(m \geq 2)$ independent gamma populations with the probability density function (pdf) given by

$$f_i(x) = \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}} x^{\alpha_i-1} e^{-\frac{x}{\beta_i}}, \quad x > 0, \quad \alpha_i, \beta_i > 0$$

for $i = 1, 2, \dots, m$. First, we develop a test based on generalized minimum chi-square for testing general linear hypothesis regarding the parameters of these m gamma populations. Then, we will specialize this test for testing equality of the m shape parameters. For this, let us consider testing the general linear hypothesis

$$H_0 : \mathbf{C}\boldsymbol{\theta} = \Phi_0 \text{ against } H_1 : \mathbf{C}\boldsymbol{\theta} \neq \Phi_0.$$

where $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1^{*'}, \boldsymbol{\theta}_2^{*'}, \dots, \boldsymbol{\theta}_m^{*'})$ with $\boldsymbol{\theta}_i^{*'} = (\alpha_i, \beta_i)$, \mathbf{C} is an $r \times 2m$ matrix of rank r and Φ

is an $r \times 1$ vector of known constants. Let $\kappa_{il} = (l-1)! \alpha_i \beta_i^l$ be the l^{th} cumulant of the i^{th} gamma population for $i = 1, 2, \dots, m$, and $l = 1, 2, 3, \dots$. Define

$$\begin{aligned}\eta_{i0} &= \kappa_{i1} = \alpha_i \beta_i \\ \eta_{il} &= \frac{\kappa_{i,l+1}}{\kappa_{il}} = l \beta_i \\ \tau_{il} &= \frac{\kappa_{il}}{\eta_{il}^l} = \frac{(l-1)!}{l^l} \alpha_i,\end{aligned}$$

for $i = 1, 2, \dots, m$, and $l = 1, 2, 3, 4, \dots$.

We formulate the following two GMC methods depending on the number of η and τ functions utilized.

Method 1: Based on two pairs of τ and η functions

Let

$$\begin{aligned}\boldsymbol{\eta}_i &= \begin{bmatrix} \tau_{i1} \\ \eta_{i1} \\ \tau_{i2} \\ \eta_{i2} \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \beta_i \\ \frac{1}{4}\alpha_i \\ 2\beta_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \\ &= \mathbf{w}^* \boldsymbol{\theta}_i^*\end{aligned}$$

with

$$\mathbf{w}^{*'} = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Now, let

$$\boldsymbol{\eta}' = (\boldsymbol{\eta}'_1, \boldsymbol{\eta}'_2, \dots, \boldsymbol{\eta}'_m)$$

and

$$\mathbf{w} = \text{diag}(\mathbf{w}^*, \mathbf{w}^*, \dots, \mathbf{w}^*).$$

This gives us a linear relationship $\boldsymbol{\eta} = \mathbf{w}\boldsymbol{\theta}$, where $\boldsymbol{\eta}$ is a $4m \times 1$ vector, \mathbf{w} is a $4m \times 2m$ matrix of known constants and $\boldsymbol{\theta}$ is a $2m \times 1$ vector of parameters. Utilizing this we will develop the GMC estimators of the parameters and develop a test for the general linear hypothesis which will be used to test hypotheses about the shape and scale parameters.

Method 2: Based on three pairs of τ and η functions

Let

$$\begin{aligned} \boldsymbol{\eta}_i &= \begin{bmatrix} \tau_{i1} \\ \eta_{i1} \\ \tau_{i2} \\ \eta_{i2} \\ \tau_{i3} \\ \eta_{i3} \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \beta_i \\ \frac{1}{4}\alpha_i \\ 2\beta_i \\ \frac{2}{27}\alpha_i \\ 3\beta_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{4} & 0 \\ 0 & 2 \\ \frac{2}{27} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \\ &= \mathbf{w}^* \boldsymbol{\theta}_i^* \end{aligned}$$

with

$$\mathbf{w}^{*'} = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & \frac{2}{27} & 0 \\ 0 & 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

Now, let

$$\boldsymbol{\eta}' = (\eta'_1, \eta'_2, \dots, \eta'_m)$$

and

$$\mathbf{w} = \text{diag}(\mathbf{w}^*, \mathbf{w}^*, \dots, \mathbf{w}^*).$$

This gives us a linear relationship $\boldsymbol{\eta} = \mathbf{w}\boldsymbol{\theta}$, where $\boldsymbol{\eta}$ is a $6m \times 1$ vector, \mathbf{w} is a $6m \times 2m$ matrix of known constants and $\boldsymbol{\theta}$ is a $2m \times 1$ vector of parameters. We utilize these relationships to obtain the GMC estimators of the parameters and to develop asymptotic tests regarding the shape and scale parameters.

The elements of \mathbf{J}_{i1} and \mathbf{J}_{i2} are as follows:

$$\mathbf{J}_{i1} = \frac{\partial(\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4})}{\partial(\mu'_{i1}, \mu'_{i2}, \mu'_{i3}, \mu'_{i4})} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2\mu'_{i1} & 1 & 0 & 0 \\ -3\mu'_{i2} + 6\mu'^2_{i1} & -3\mu'_{i1} & 1 & 0 \\ -4\mu'_{i3} + 24\mu'_{i2}\mu'_{i1} - 24\mu'^3_{i1} & -6\mu'_{i2} + 12\mu'_{i1} & -4\mu'_{i1} & 1 \end{bmatrix}$$

$$\mathbf{J}_{i2} = \frac{\partial((\tau_{i1}, \eta_{i1}, \tau_{i2}, \eta_{i2}, \tau_{i3}, \eta_{i3}))}{\partial(\kappa_{i1}, \kappa_{i2}, \kappa_{i3}, \kappa_{i4})} = \begin{bmatrix} \frac{2\kappa_{i1}}{\kappa_{i2}} & -\frac{\kappa_{i1}^2}{\kappa_{i2}^2} & 0 & 0 \\ -\frac{\kappa_{i2}}{\kappa_{i1}^2} & \frac{1}{\kappa_{i1}} & 0 & \\ 0 & \frac{3\kappa_{i2}^2}{\kappa_{i3}^2} & -\frac{2\kappa_{i2}^3}{\kappa_{i3}^3} & 0 \\ 0 & -\frac{\kappa_{i3}}{\kappa_{i2}^2} & \frac{1}{\kappa_{i2}} & 0 \\ 0 & 0 & \frac{4\kappa_{i3}^3}{\kappa_{i4}^3} & -\frac{3\kappa_{i3}^4}{\kappa_{i4}^3} \\ 0 & 0 & \frac{\kappa_{i4}}{\kappa_{i3}^2} & \frac{1}{\kappa_{i3}} \end{bmatrix}.$$

Test Statistics for testing H_0 and its asymptotic distribution

To develop the test statistic based on the GMC procedure, we consider the quadratic form

$$Q = (\mathbf{h} - \mathbf{w}\boldsymbol{\theta})'(\hat{\boldsymbol{\Sigma}})^{-1}(\mathbf{h} - \mathbf{w}\boldsymbol{\theta})$$

and minimize it under no restrictions and under H_0 . Let $\hat{\boldsymbol{\theta}}$ be the estimator of $\boldsymbol{\theta}$ under no restriction and $\tilde{\boldsymbol{\theta}}$ be its estimator under H_0 . Then, it can be seen that

$$\hat{\boldsymbol{\theta}} = (\mathbf{w}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{w})^{-1}(\mathbf{w}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{h}),$$

and

$$\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - (\mathbf{w}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{w})^{-1}\mathbf{C}'(\mathbf{C}(\mathbf{w}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{w})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\hat{\boldsymbol{\theta}} - \boldsymbol{\Phi}_0).$$

Let Q_0 be the minimum value of Q under H_0 and Q_1 be its value under no restrictions. Then, the test statistics for testing H_0 is $T = Q_0 - Q_1$ and, asymptotically under H_0 , $T \sim \chi_r^2$ where $r = \text{rank}(\mathbf{C})$.

3. Tests for parameters of three gamma populations based on Method 1

Here, we specialize the above general test to test (i) equality of the shape parameters, and (ii) equality of the shape as well as the scale parameters of three gamma populations based on Method 1 where we use two pairs of τ and η functions.

(i) Equality of Shape parameters

To test the equality of the shape parameters of three gamma populations using Method 1, we choose the coefficient matrix \mathbf{C} as

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

With this choice of \mathbf{C} , we can compute the test statistic $T = Q_0 - Q_1$ which will have an asymptotic χ_2^2 distribution.

(ii) Equality of Shape and Scale parameters

To test the hypothesis of equality of the shape and the scale parameters of three gamma populations using Method 1, we choose the \mathbf{C} matrix as

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

With this choice of \mathbf{C} , we can compute the test statistic $T = Q_0 - Q_1$ which will have an asymptotic χ_4^2 distribution.

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