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Likelihood ratio tests in two gamma populations for equality of shape parameters

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Abstract

The introduction of shape parameters into the statistical literature opened new areas of research and allowed statisticians to use models that produced better fits to experimental data. The Weibull and gamma families are prime examples wherein shape parameters produce more reliable statistical models than standard exponential models in lifetime studies. In the presence of many gamma-populations, one may test equality (or homogeneity) of shape parameters across a collection of independent populations. In this paper we develop standard asymptotic tests for testing the equality of shape parameters of gamma distributions using the log-likelihood ratio (LRT) test statistic. Other tests are given that summarize test hypotheses on the shape parameter of a single gamma distribution. We numerically investigate the performances of these tests and find that in large sample sizes that the distribution of the log-likelihood ratio test statistic converges nicely to that of a chi-square distribution.

KEYWORDS: Beta distribution, Dirichlet distribution, log-likelihood ratio test, and Stochastic ordering.

JEL Classification Codes: C12, C15, C16, C88

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1 Introduction

The purpose of this article is the development of some hypothesis tests for testing the equality of shape parameters of two-parameter gamma distributions from two independent populations. These tests are developed under the constraint that the actual Type I error rate should not exceed the prescribed level of significance, (α), even for small sample sizes. In what follows, we first explore two conservative tests for the shape parameter of a gamma distribution. For every testing problem, we numerically compute the Type I error rate via Monte Carlo simulation and show that in most cases the simulated Type I error rate is under control. To compare several independent gamma populations, we develop a test by using the generalized likelihood ratio test and verify the traditional asymptotic approach via the log-likelihood ratio test statistic.

1.1 Applications of the Gamma–Family

The gamma distribution has many practical applications across many spectra of human endeavors and scientific discoveries. In human survival, the family is one of wide applicability in that its hazard function can be decreasing (DFR), constant, or increasing (IFR): the three common features of most empirically derived hazard functions, also known as the bathtub curve. One can generate a bathtub curve model by connecting three separate gamma distributions piecewise via change points. A random variable X that follows gamma law has a density function is given by

$$f\left(x;\kappa,\beta\right)=\frac{x^{\kappa-1}\exp(-x/\beta)}{\Gamma\left(\kappa\right)\beta^{\kappa}}\times I_{(0,\infty)}\left(x\right) \tag{1}$$

The shape parameter, κ, is especially interesting to survival analysts since the gamma hazard function is decreasing, constant, or increasing according to the trichotomy of $\kappa-1$. We denote this distribution property by $X\sim\mathcal{G}(\kappa,\ \beta)$. The mean and variance of X are well known to be $E(Y)=\kappa\beta$ and $V(Y)=\kappa\beta^2$. Epidemiologists, engineers, and other scientists note that many experimental settings lead to a random variables in which the coefficient of variation is a deterministic constant. If the random variable follows the gamma law then knowledge of the coefficient of variation implies knowledge of κ. The shape parameter is a recurring model in renewal theory for times to failure, remission, etc., that follow from a set of formal assumptions.

Special cases of random variables having gamma distributions, include chi-square random variables for which $\kappa = \nu/2$, where the positive integer, ν, is the number of degrees of freedom of the chi-square random variable, and $\beta = 2$. Another special case of the Gamma–family is given by the Erlang family in which $\kappa = n$ where n is a positive integer. Its importance is seen because a finite series can be given for the cumulative distribution of the random variable. In general this cannot be done for all members of the Gamma–family of random variables. The most widely used special case of the Gamma–family occurs when $\kappa = 1$ and is known as the exponential family. It is well known that the exponential distribution has a constant failure rate distribution. The value of $\kappa = 1$ separates the parameter space Ω into two disjoint regions. The Erlang distribution is often considered as the sum of n independent and identically distributed exponential random variables with mean time to failure of β. Another well known case of the (IFR) Gamma–family occurs when $\kappa = \frac{3}{2}$ which is called Maxwell's distribution. When $\kappa = 2$, we get the length-biased version of the exponential family which is quite important in sampling.

1.2 Competing Risks

Consider a homogeneous population of n individuals with lives that are at risk to a diseases or competing causes of death, such as cardiovascular disease, cancer, diabetes, etc. (see Pintille [19] or Crowder [5]).

Let $x_\ell,\ \ell = 1, \dots, n$ be the observed lifetime of an individual and let Δ_ℓ be the indicator which specifies the cause of death for the ℓ^{th} individual. The random variable Δ_ℓ has support on the set $\{1, \dots, p\}$. If the person expires due to the i^{th} disease, we have:

$$\Delta_\ell = i \quad \text{and} \quad \Pr(\Delta_\ell = i) = \pi_i. \tag{2}$$

The natural restriction is that $\sum_{i=1}^{p} \pi_i = 1$. It follows that the conditional distribution of an individual's lifetime due to the i^{th} cause is denoted by $x \mid \Delta = i$ and has the same pdf as the i^{th} family.

$$\begin{aligned} f\left(x, \Delta = i\right) &= f\left(x \mid \Delta_\ell = i\right) \Pr\left(\Delta_\ell = i\right) \\ &= f_i\left(x\right) \pi_i. \end{aligned} \tag{3}$$

Using the previous expression, we can write a general expression for the likelihood of n independent deaths due to a competing causes as

$$L = \prod_{i=1}^{p} \left[\prod_{j=1}^{n_i} f_i\left(x_{ij}\right) \pi_j \right] \tag{4}$$

where x_{ij} is the j^{th} lifetime due to the i^{th} cause of death and n_i is the number of deaths due to the i^{th} cause. Within the context of competing risks, we introduce multiple independent Gamma distributions each representing the length of life ended by the disease.

2 Foundations

For each i, $i = 1, 2, \ldots, p$, suppose x_{ij} are n_i i.i.d. gamma random variables with shape parameter κ_i and scale parameter β_i for each $j = 1, 2, \ldots, n_i$. The joint density function for the sample obtained from the i^{th} population is given by

$$f\left(x_{i1}, \ldots, x_{in_i} \mid \kappa_i, \beta_i\right) = \frac{\tilde{x}_i^{n_i(\kappa_i - 1)} \exp(-\frac{n_i \bar{x}_i}{\beta_i})}{[\beta_i^{\kappa_i} \Gamma(\kappa_i)]^{n_i}} \tag{5}$$

The arithmetic and geometric means of this random sample are denoted respectively by $\bar{x}_i$ and $\tilde{x}_i$. Let $w_i = \tilde{x}_i / \bar{x}_i$ denote the ratio of the sample geometric mean to the sample arithmetic mean and this ratio by Jensen's inequality, is $0 \le w_i \le 1$. Following Bhaumik et al.[4] define $R_i\left(\boldsymbol{x}_i\right) = -n_i \ln\left(w_i\right)$ as the natural logarithm of their reciprocal, where $\boldsymbol{x}_i$ is the vector of observed values from the sample of size n_i obtained from the i^{th} population. Thus from the joint distribution, we obtain the likelihood function as

$$L_i\left(\kappa_i, \beta_i \mid \boldsymbol{x}_i\right) = \frac{\tilde{x}_i^{n_i(\kappa_i - 1)} \exp(-\frac{n_i \bar{x}_i}{\beta_i})}{[\beta_i^{\kappa_i} \Gamma(\kappa_i)]^{n_i}}. \tag{6}$$

The subsequent log-likelihood, for each population, (i.e., $i = 1, \ldots, a$) is given by

$$\begin{aligned}
\mathcal{L}_i\left(\kappa_i, \beta_i \mid \boldsymbol{x}_i\right) &= \ln\left\{L_i\left(\kappa_i, \beta_i \mid \boldsymbol{x}_i\right)\right\} \\
&= n_i\left(\kappa_i - 1\right)\ln\left(\tilde{x}_i\right) - \frac{n_i \bar{x}_i}{\beta_i} \qquad (7)\\
&\quad - n_i \kappa_i \ln(\beta_i) - n_i \ln\left[\Gamma(\kappa_i)\right].
\end{aligned}$$

By methods of the calculus, one can solve for the maximum likelihood estimators of κ_i and β_i by solving the following system of equations numerically:

- $\ln\left(w_i\right) = \psi\left(\hat{\kappa}_i\right) - \ln\left(\hat{\kappa}_i\right)$.
  The solution obtained numerically for $\hat{\kappa}_i$ is the MLE of κ_i where $\psi(x)$ is the digamma function. The MLE of κ, denoted as $\widehat{kappa}_i$, is substituted into the following equation to obtain the corresponding MLE for β_i.
- $\hat{\kappa}_i \hat{\beta}_i = \bar{x}_i$.

Evaluation of the log-likelihood function at the maximum likelihood estimators, yields the following expression

$$\begin{aligned}
\mathcal{L}_i\left(\hat{\kappa}_i, \hat{\beta}_i \mid \boldsymbol{x}_i\right) &= n_i \hat{\kappa}_i \ln\left(w_i\right) - n_i \hat{\kappa}_i - n_i \ln\left[\Gamma\left(\hat{\kappa}_i\right)\right] \qquad (8)\\
&\quad + n_i \hat{\kappa}_i \ln(\hat{\kappa}_i) - n_i \ln\left(\tilde{x}_i\right)
\end{aligned}$$

As $\hat{\kappa}_i$ depends upon the data only through the statistics w_i, then it becomes clear that the estimated log-likelihood depends upon the data only through w_i and $\tilde{x}_i$ thus its distribution depends only on the joint distribution of w_i and $\tilde{x}_i$.

Assuming that the shapes parameters are equal to a common value $H_0 : \kappa_0 = \kappa_1 = \kappa_2 = \ldots = \kappa_p$, then one can write the likelihood function of $\kappa_0, \beta_1, \ldots, \beta_p$ as

$$L_0\left(\kappa_0, \beta_1 \ldots \beta_p \mid \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_p\right) = \frac{\Pi_{i=1}^{p} \tilde{x}_i^{n_i(\kappa_0 - 1)} \exp\left(-\sum_{i=1}^{p} \frac{n_i \bar{x}_i}{\beta_i}\right)}{\Pi_{i=1}^{p}\left[\beta_i^{\kappa_0}\Gamma(\kappa_0)\right]^{n_i}}. \qquad (9)$$

It then follows that

$$\begin{aligned}\mathcal{L}_0\left(\kappa_0, \beta_1 \ldots \beta_p \mid \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_p\right) &= (\kappa_0 - 1)\sum_{i=1}^{p} n_i \ln\left(\tilde{x}_i\right) - \sum_{i=1}^{p} \frac{n_i \bar{x}_i}{\beta_i} \\ &- \kappa_0 \sum_{i=1}^{p} n_i \ln\left(\beta_i\right) - n_0 \ln\left[\Gamma(\kappa_0)\right] \qquad (10)\end{aligned}$$

where $n_0 = \sum_{i=1}^{p} n_i$.

Taking the partial derivative of (10) with respect to κ_0 produces:

$$\frac{\partial \mathcal{L}_0}{\partial \kappa_0} = \sum_{i=1}^{p} n_i \ln\left(\tilde{x}_i\right) - \sum_{i=1}^{p} n_i \ln\left(\beta_i\right) - n_0 \psi(\kappa_0) \qquad (11)$$

where $\psi(x)$ is Euler's digamma function evaluated at the positive real number, x. Likewise,

$$\frac{\partial \mathcal{L}_0}{\partial \beta_i} = \frac{n_i \bar{x}_i}{\beta_i^2} - \frac{\kappa_0 n_i}{\beta_i} \qquad (12)$$

for each $i = 1, \ldots, p$. Setting the equations in (12) equal to zero we obtain the following p equations in $p+1$ unknowns.

$$\beta_i = \bar{x}_i / \kappa_0. \qquad (13)$$

Substituting these solutions into (11) one obtains,

$$\frac{\partial \mathcal{L}_0}{\partial \kappa_0} = \sum_{i=1}^{p} n_i \ln\left(w_i\right) + n_0 \ln\left(\kappa_0\right) - n_0 \psi(\kappa_0) \qquad (14)$$

Setting the partial derivative equal to zero we obtain,

$$\begin{aligned}\sum_{i=1}^{p} n_i \ln\left(w_i\right) &= n_0 \psi(\kappa_0) - n_0 \ln\left(\kappa_0\right) \\ \ln\left(\Pi_{i=1}^{p} w_i^{c_i}\right) &= \psi(\kappa_0) - \ln\left(\kappa_0\right) \qquad (15)\end{aligned}$$

where $c_i = n_i / n_0$. Thus the maximum likelihood estimator of κ_0 satisfies the same equation (15) as the ones for random samples taken from individual populations where data are involved through the weighted geometric mean of the ratios of sample geometric to sample arithmetic means in samples of size n_i. In this vein, define

$$W = \Pi_{i=1}^{p} w_i^{c_i}. \tag{16}$$

Thus the log-likelihood evaluated at the maximum likelihood estimators assuming the null hypothesis is true becomes

$$\begin{aligned}\mathcal{L}_0\left(\hat{\kappa}_0, \hat{\beta}_1 \ldots \hat{\beta}_p \mid \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_p\right) &= n_0\hat{\kappa}_0 \ln\left(W\right) - n_0\hat{\kappa}_0 + n_0\hat{\kappa}_0 \ln\left(\hat{\kappa}_0\right) \\ &\quad -n_0 \ln\left[\Gamma\left(\hat{\kappa}_0\right)\right] - \sum_{i=1}^{p} n_i \ln\left(\tilde{x}_i\right) \quad (17)\end{aligned}$$

The likelihood ratio test is found by computing the following:

$$\Lambda\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\right) = \frac{L_0}{\Pi_{i=1}^{p} L_i}. \tag{18}$$

It follows that the log-likelihood test reduces under a logarithmic transformation to

$$\begin{aligned}\ln\left(\Lambda\right) &= \mathcal{L}_0 - \sum_{i=1}^{p} \mathcal{L}_i. \\ &= \sum_{i=1}^{p} n_i\left[\hat{\kappa}_0 - \hat{\kappa}_i\right] \ln\left(w_i\right) + \sum_{i=1}^{p} n_i\left[\hat{\kappa}_0 \ln\left(\hat{\kappa}_0\right) - \hat{\kappa}_i \ln\left(\hat{\kappa}_i\right)\right] \\ &- \sum_{i=1}^{p} n_i\left[\hat{\kappa}_0 - \hat{\kappa}_i\right] - \sum_{i=1}^{p} n_i\left\{\ln\left[\Gamma\left(\hat{\kappa}_0\right)\right] - \ln\left[\Gamma\left(\hat{\kappa}_i\right)\right]\right\} \quad (19)\end{aligned}$$

3 Testing hypotheses on a single shape parameter

In this section, we present hypothesis tests on the shape parameter of a gamma distribution assuming the scale parameter is unknown. Keating et al. [14] provide an optimal test to test hypotheses on the values of the shape parameter in a single population. The test statistic, $\bar{X}$, alone cannot be used to construct a test statistic for testing the shape parameter as the scale parameter is involved in its distribution. However, the distribution of w depends only on κ and not on β. This random variable, w follows naturally from the generalized likelihood-ratio test. Keating et al. [14] constructed this uniformly most powerful unbiased (UMPU) test for κ based on the ratio of the geometric to arithmetic sample means, w, only by expressing the density function of w in powers of $-\ln(w)$. This representation is Glaser's series expansion [9] of the distribution of w. Keating et al. [14] noted that Glaser's expression yields a conservative radius of convergence for the series, which is known to converge for all w in the closed-bounded interval $[\exp(-2\pi/n), 1]$. This condition is problematic whenever the alternative hypothesis is left-sided and occurs frequently when one may test the null of exponentiality against a DFR alternative.

In the material that follows we recount the research of Bhaumik et al. [4] on approximations to distributions of test statistics in small sample sizes. These results follow from an exact expression of the test statistic as a product of independent beta random variables but with different shape parameters although its power is not optimal as it does not follow from the generalized likelihood ratio test.

3.1 Right–tail alternatives

Let us first investigate the problem for:

$$\begin{aligned} H_{01} : \kappa &= \kappa_0 \\ &vs \\ H_{a1} : \kappa &> \kappa_0 \end{aligned} \tag{20}$$

They wanted a simple test that controls the Type I error rate, α, for even very small values of n. The procedure involves construction of a random variable Z which is stochastically larger than w^n (denote this ordering by $Z \succ w^n$) and develop a conservative test based on Z. In this section, Z, conservative test statistic, follows a beta distribution rather than a product of independent beta random variables having different shape parameters. Denote the $100p^{th}$ percentile of a beta distribution with parameters ξ and δ by $b_p(\xi,\ \delta)$; so that $\Pr\left(Y < b_p(\xi,\ \delta)\right) = p$ where Y follows the beta distribution with shape parameters ξ and δ, respectively. The test that follows is easy to implement for H_{01}. The result is presented below in the form of a theorem.

Theorem 1 Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution as in (1) with shape parameter κ and scale parameter β. Let w be the ratio of the sample geometric to the arithmetic sample mean. A conservative rejection region in testing the hypothesis H_{01} against the alternative H_{a1} in 20, is given by :

$$\left\{ \boldsymbol{x} : w^n > b_{1-\alpha}\left(\kappa_0,\ \frac{n-1}{2}\right)\right\} \tag{21}$$

where $b_p(\xi,\ \delta)$ is the $100p^{th}$ percentile point of a beta distribution with parameters ξ and δ.

Proof. Glaser [9] proved that the distribution of w^n is distributed as a product of $(n-1)$ independent beta distributions, i.e.

$$w^n \sim \prod_{i=1}^{n-1} Z_i \tag{22}$$

where $Z_i \sim \mathcal{B}(\kappa, \frac{i}{n})$, $i = 1, 2, \cdots, n-1$. Note that $\mathcal{B}(\kappa, \frac{i}{n})$ is stochastically increasing in κ. It follows that we can construct a random variable $Z_1^* \sim \mathcal{B}(\kappa + \frac{2}{n}, \frac{1}{n})$ that is stochastically larger than Z_1. Using the succession of beta's result, see Bhaumik et al. [4], we have

$$Z_1 Z_2 \prec Z_1^* Z_2 = Z_2^*,\ \text{where } Z_2^* \sim \mathcal{B}(\kappa, \frac{1}{n} + \frac{2}{n}). \tag{23}$$

By the induction principle, we can construct a random variable Z which is stochastically larger than w^n, i.e. $w^n \prec Z$, where $Z \sim \mathcal{B}(\kappa_0, \frac{n-1}{2})$. Thus, $\Pr\left[w^n > b_{1-\alpha}(\kappa_0, \frac{n-1}{2})\right] < \Pr\left[Z > b_{1-\alpha}(\kappa_0, \frac{n-1}{2})\right] = \alpha$. Hence, the test is conservative.

Example 1 Consider the IFR example given in Keating et al. [14] where they enumerate 29 operating hours of air-conditioning systems in aircraft 7909. The sample geometric mean is given by $\tilde{x} = 60.15$ hours and arithmetic mean of $\bar{x} = 83.52$ hours. The resulting value of $u = 0.7203$. $b_{1-\alpha}(\kappa_0, \frac{n-1}{2})) = b_{.95}(1, 14) = 0.00365710258483887$,
$u = 0.7203,\ [b_{0.95}(1, 14)]^{1/29} = 0.82402$

3.2 Left tail alternatives

Let us now consider another hypothesis $H_{02} : \kappa = \kappa_0$ against the alternative $H_{a2} : \kappa < \kappa_0$. The approach used in Theorem 1 would not produce a conservative left-sided test. By contrast, to construct a conservative test for H_{02}, we should look for a random variable Z which is stochastically smaller than w^n.

Theorem 2 Let $X_1, X_2, \ldots, X_n$ be a random sample from a gamma distribution with shape parameter κ and scale parameter β, and w be the ratio of the geometric mean to the arithmetic mean. Let Y_i's $i = 1, 2, \cdots, n-1$ be $n-1$ independent and identically distributed beta random variables each with parameters κ and $\frac{n-1}{n}$ and

$$Z = \prod_{i=1}^{n-1} Y_i.$$

Denote the $(\alpha)100\%$th percentile point of the distribution of Z by $Z_\alpha(\kappa, \frac{n-1}{n})$. For testing the hypothesis H_{02} against the alternative H_{a2} there exists a conservative test given by: reject H_{02} if $w^n < Z_\alpha(\kappa, \frac{n-1}{n})$.

Proof. In Theorem 1 we stated that $w^n \sim \prod_{i=1}^{n-1} Z_i$, where $Z_i \sim \mathcal{B}(\kappa, \frac{i}{n})$. Note that each $\mathcal{B}\left(\kappa, \frac{i}{n}\right)$ is stochastically decreasing in i. As $Y_i \sim \mathcal{B}\left(\kappa, \frac{n-1}{n}\right)$, $i = 1, \ldots, n-1$, hence $Y_i \prec Z_i$, for all $i = 1, 2, \ldots, n-1$. Thus $Z = \prod_{i=1}^{n-1} Y_i$ is stochastically smaller than w^n.

Derivation of the exact distribution of Z and determination of its percentile point c may be mathematically challenging but our approach will lose its simplistic appeal. Instead, we determine c by Monte Carlo simulation. Exploiting the fact that Y_i's are independently distributed, we can further simplify the computation as follows.

$$\begin{aligned} \Pr(w^n < c) \;\leq\; \Pr(Z < c) &= 1 - \Pr(Z > c) \\ &\leq 1 - \prod_{i=1}^{n-1} \Pr\left[\mathcal{B}\left(\kappa, \frac{i-1}{n}\right) > c^{\frac{1}{n-1}}\right] \\ &= 1 - \left[\Pr(\mathcal{B}(\kappa, \frac{n-1}{n}) > c^{\frac{1}{n-1}})\right]^{n-1} \quad (24) \end{aligned}$$

The middle inequality in (24) is obtained from the fact that Y_i's are independently and identically distributed. Determination of c is simplified as the extreme right side of (24) is based on a single beta random variable. In order to determine the cut-off point c, we equate the extreme right side of (24) to the given nominal level of significance α. In turn this test becomes more conservative.

3.3 Simulations

In order to evaluate the performance of the proposed test for H_{01} we have numerically computed the Type I error rate (based on $100,000$ samples) for a wide range of κ starting from 1 to 10 and for $n = 2, \cdots, 10$. For smaller values of $\kappa(\leq 5)$ we observe that the simulated Type I error rates are always between 0.03 and 0.05 when n is not more than 5. For larger values of $\kappa(> 5)$ we obtain similar results for $n = 2, \ldots, 10$.

We evaluate the performance of H_{02} by the same way as mentioned before. When c is determined numerically based on the distribution of Z, the simulated Type 1 error rates match with those described before. But when c is determined from the extreme right side of (24), we notice that the test performs reasonably well for very small values of n (< 5) and for a wide range of κ between 0.25 to 30.

Note that tests proposed in Theorems 1 and 2 are conservative and their performances are not satisfactory for large values of n. In this context, we evaluated a test for κ using Theorem 2. We compute the mean and variance of $R_n = -n\ln(w)$ using the Delta method. The constants associated with a chi-square approximation, c and ν, are determined from the following equations:

$$\begin{aligned} 2n\kappa_0 E(R_n) &= c\nu \\ 2n\kappa_0 Var(R_n) &= 2c^2\nu. \end{aligned} \quad (25)$$

The test is to reject H_{01} if $2n\kappa_0 R_n < c\chi^2_{\nu,1-\alpha}$, and H_{02} if $2n\kappa_0 R_n > c\chi^2_{\nu,\alpha}$. An extensive simulation study based on $100,000$ samples indicates that the overall performance of these tests is extremely well in terms of simulated Type I error rate for all values of $n = 1, \cdots, 30$ and

$\kappa > 1$. Even for $\kappa < 1$ and for small values of n (< 5) the simulated Type I error rate did not exceed 0.065 when α was fixed at .05.

4 Comparisons of two gamma populations

In this section we assume that we have $p = 2$ independent gamma populations. We would like to compare their shape parameters. Assume that we have n_i independent observations from each of two independent gamma populations, i.e. $x_{ij} \sim \mathcal{G}(\kappa_i,\ \beta_i)$ $j = 1, 2, \ldots, n_i$ and $i = 1, 2$. Our hypothesis of interest is $H_o : \kappa_1 = \kappa_2$ against the alternative hypothesis, $H_a : \kappa_1 \neq \kappa_2$. In engineering, when components of a product are made in different plants for the purpose of using them in the same system, it is essential to check whether they are equivalent. In a study of this type, generally shape parameters govern the decision as data are often rescaled, so that one can often assume equality of the scale parameters, $\beta_1 = \beta_2 = 1$. In the first part of this section, we develop a test for H_o under the assumption that $x_{ij} \sim \mathcal{G}(\kappa_i,\ \beta)$. Let, $Z = \sum_{i=1}^{2} \sum_{j=1}^{n_i} x_{ij}$ and $z_{ij} = \frac{x_{ij}}{Z}$. The joint distribution of z_{ij}'s is Dirichlet with the following density function:

$$
\begin{aligned}
f(z_{11}, \cdots, z_{2n_2}) &= \frac{\Gamma(n_1\kappa_1 + n_2\kappa_2)}{(\Gamma(\kappa_1))^{n_1}(\Gamma(\kappa_2))^{n_2}} \\
&\times \ z_{11}^{(\kappa_1-1)} \cdots z_{1n_1}^{(\kappa_1-1)} z_{21}^{(\kappa_2-1)} \cdots z_{2n_2}^{(\kappa_2-1)}. \qquad (26)
\end{aligned}
$$

Marshall and Olkin [17] called this distribution as Dirichlet Type III distribution. We now proceed to construct an unbiased test (in the sense of Schur convexity) for H_{03} using z_{ij}'s and their joint distribution as defined above. Let us state the result in the form of a theorem.

Theorem 3 Let $x_{i_1}, x_{i_2}, \cdots, x_{i_{n_i}}$ be a random sample from the i^{th} gamma population with shape parameter κ_i and scale parameter β, where $i = 1, 2$. Denote the $100(\alpha)^{th}$ percentile point of a Dirichlet's distribution defined in (26) when all the κ's are equal by D_α. For testing the hypothesis H_{03} against the alternative H_{a3} there exists a unbiased test in the sense of Schur convexity given by: reject H_{03} if

$$Q(z_{11}, \cdots, z_{2n_2}) < D_\alpha$$

where,

$$Q(z_{11}, \cdots, z_{2n_2}) = \prod_{i=1}^{2} \prod_{j=1}^{n_i} z_{ij}.$$

Proof. $Q(z_{11}, \cdots, z_{2n_2})$ is a Schur-concave function of random variables $z_{1_1}, \cdots, z_{2_{n_2}}$. Note that $\sum_{i=1}^{2} \sum_{j=1}^{n_i} z_{ij} = 1$. It can be proven that the indicator function $I_{\{Q(z_{11},\cdots,z_{2n_2})<D_\alpha\}}$ is Schur-convex. Dirichlet's Type III distributions parameterized by the vector κ_1 and κ_2 has the property that expectation of a Schur-convex function leads to a Schur-convex function of κ_1 and κ_2 (see Marshall and Olkin [17], Chapter 11). Hence, $\Psi(\kappa_1, \kappa_2) = E(I_{\{Q(z_{11},\cdots,z_{2n_2})<D_\alpha\}})$ is a Schur-convex function of κ_1, κ_2. Thus under H_{03}, $\Psi(\kappa_1, \kappa_2)$ takes its minimum value as (κ, κ) is majorized by (κ_1, κ_2). But, the value of (κ_1, κ_2) is (κ, κ) under H_{03}. Therefore, the proposed test is unbiased in the sense of Schur-convexity.

In order to implement the result stated in Theorem 3, we need the common value of κ say, κ_0 under $H_{03} = \kappa_1 = \kappa_2 = \kappa_0$. We will call κ_0 as the target value. The critical value D_α of Theorem 3 depends on the target value κ_0. The power of the test at (κ_1, κ_2) is always greater than α whenever $\sum_{i=1}^{2} n_i \kappa_i = \kappa_0 \sum_{i=1}^{2} n_i$.

In case κ_0 is not specified we should look for a test that does not depend on the target value. Without any loss of generality, let n_1 be the minimum of $(n_1, \cdots, n_p)$. Using the second part of Theorem 3, we can construct a test

$$F = \frac{n_1 R_{n_1}}{n_2 R_{n_2}}$$

that follows, under the truth of H_{03}, a central F-distribution with df's $n_1 - 1$ and $n_2 - 1$. Note that to construct this F we do not need the scale parameters of the distributions to be equal. Neither the construction nor the distribution of F depends on any parameters under the null hypothesis.

5 The Likelihood Function

Recall the competing risks example discussed in Section 1.2 where multiple independent Gamma populations are involved. In this section we discuss the maximum likelihood estimation procedure for estimating model parameters under the assumption that each gamma population has its own scale and shape parameters. Thus we have $\kappa_1, \kappa_2, \cdots, \kappa_p$ shape parameters and $\beta_1, \beta_2, \cdots, \beta_p$ scale parameters. In Sections 5.1 and 5.2 we discuss some special cases regarding the equality of these model parameters and provide a sketch of estimation. The joint density function of the gamma random variables in the i^{th} random sample of size n_i, $Y_1, \ldots, Y_{n_i}$, obtained from this gamma distribution is given by:

$$f_i(y_1, \ldots, y_{n_i}; \kappa_i, \beta_i) = \prod_{j=1}^{n_i} \frac{y_j^{\kappa_i - 1} e^{-y_j/\beta_i}}{\Gamma(\kappa_i) \beta_i^{\kappa_i}} I_{(0,\infty)}\left(y_{(1)_i}\right) \tag{27}$$

and $j = 1, \ldots, n_i$ for each $i = 1, \ldots, p$. Recall that when this joint density function of the i^{th} gamma family is viewed as a function of the parameters κ_i and β_i given the data, $\mathbf{y}_i$ (the vector of observations), the function is the likelihood function, $\mathrm{L}_i(\kappa_i, \beta_i | \mathbf{y})$, by Fisher. The likelihood function of the i^{th}gamma family is given by

$$\begin{aligned} \mathrm{L}_i(\kappa_i, \beta_i | \mathbf{y}_i) &= f_i(\mathbf{y}_i; \kappa_i, \beta_i) \\ &= \frac{\tilde{y}_i^{n_i(\kappa_i - 1)} e^{-n_i \bar{y}_i/\beta_i}}{[\Gamma(\kappa_i)]^{n_i} \ \beta_i^{n_i \kappa_i}} I_{(0,\infty)}\left(y_{(1)_i}\right), \end{aligned} \tag{28}$$

where $\tilde{y}_i$ is the geometric mean of the i^{th} random sample observations and $\bar{y}_i$ is their corresponding arithmetic mean the i^{th} sample. These statistics form a pair of sufficient statistics for the i^{th} gamma family.

$$\tilde{y}_i = \left(\prod_{j=1}^{n_i} y_{ij}\right)^{1/n_i} \qquad \bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} \qquad u_i = \frac{\tilde{y}_i}{\bar{y}_i}$$

Let w_i signify the ratio of geometric to arithmetic mean in the i^{th} gamma family, $i = 1, \ldots, p$, where p is the number of families.

The joint density function of the gamma random variables for all data from the p populations with respective sample sizes $n_1, \ldots, n_p$, obtained from these gamma distributions is given by:

$$\begin{aligned} f_0\left(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_p; \boldsymbol{\kappa}, \boldsymbol{\beta}\right) &= \prod_{i=1}^{p} f_i\left(\boldsymbol{y}_i\right) \\ &= \prod_{i=1}^{p} \prod_{j=1}^{n_i} \frac{y_{ij}^{\kappa_i-1} e^{-y_{ij}/\beta_i}}{\Gamma\left(\kappa_i\right) \beta_i^{\kappa_i}} I_{(0,\infty)}\left(\min y_{ij}\right) \end{aligned} \tag{29}$$

Recall that when the joint density of the i^{th} gamma family is viewed as a function of the parameters κ_i and β_i given the data, $\boldsymbol{y}_i$ (the vector of n_i observations), the function is the likelihood function, $\mathrm{L}_i\left(\kappa_i, \beta_i | \boldsymbol{y}\right)$, by Fisher. Let $\boldsymbol{\kappa}$ denote the vector of p shape-parameters. The likelihood function of all the information contained in the p gamma-families is given by

$$\begin{aligned} \mathrm{L}_0\left(\boldsymbol{\kappa}, \boldsymbol{\beta} | \boldsymbol{y}_1 \ldots \boldsymbol{y}_p\right) &= \prod_{i=1}^{p} \mathrm{L}_i\left(\kappa_i, \beta_i | \boldsymbol{y}_i\right) \\ &= \prod_{i=1}^{p} \frac{\tilde{y}_i^{n_i(\kappa_i-1)} e^{-n_i \bar{y}_i/\beta_i}}{\left[\Gamma\left(\kappa_i\right)\right]^{n_i} \ \beta_i^{n_i\kappa_i}} I_{(0,\infty)}\left(\min y_{ij}\right) \\ &= \frac{\left[\prod_{i=1}^{p} \tilde{y}_i^{\,n_i(\kappa_i-1)}\right] \exp\left(-\sum_{i=1}^{p} n_i \bar{y}_i/\beta_i\right)}{\left[\prod_{i=1}^{p} \Gamma\left(\kappa_i\right)\right]^{n_i} \ \prod_{i=1}^{p} \beta_i^{n_i\kappa_i}} I_{(0,\infty)}\left(\min y_{ij}\right) \end{aligned} \tag{30}$$

In exponential families, it is more convenient to deal with the natural logarithm of likelihood function. Let the log-likelihood function associated with the i^{th} population be denoted by $\mathcal{L}_i\left(\kappa_i, \beta_i | \boldsymbol{y}_i\right)$ which for each $i, \ i = 1, \ldots, p$ is:

$$\mathcal{L}_i(\kappa_i, \beta_i | \boldsymbol{y}_i) = \ln\left(\mathrm{L}_i(\kappa_i, \beta_i | \boldsymbol{y})\right).$$

It follows that the log-likelihood function for all the data is given by

$$\begin{aligned} \mathcal{L}_0\left(\boldsymbol{\kappa}, \boldsymbol{\beta} | \boldsymbol{y}_1, \ldots, \boldsymbol{y}_p\right) &= \sum_{i=1}^{p} \left(n_i(\kappa_i - 1)\right) \ln\left(\tilde{y}_i\right) - \sum_{i=1}^{p} \left(n_i\right) \frac{\bar{y}_i}{\beta_i} \\ &- \sum_{i=1}^{p} n_i \left\{\ln\left[\Gamma\left(\kappa_i\right)\right]\right\} - \sum_{i=1}^{p} \left(n_i \kappa_i\right) \ln\left(\beta_i\right) \end{aligned}$$

5.1 Identically Distributed: Common Shape and Scale

The first question most would want to resolve is whether he or she is dealing with one or many populations. Let us pose the concept the following null hypothesis:

$$H_0 : \kappa_1 = \kappa_2 = \cdots = \kappa_p \text{ and } \beta_1 = \beta_2 = \cdots = \beta_p$$

versus

$$\begin{aligned} H_1 : \quad & \kappa_m \neq \kappa_t \text{ for some positive integers, } m \text{ and } t, m \neq t \\ \text{or} \quad & \beta_m \neq \beta_t \text{ for some positive integers, } m \text{ and } t, m \neq t \end{aligned}$$

Under these assumptions, the log-likelihood equations simplifies to:

$$\begin{aligned} \mathcal{L}_0 \left(\kappa, \beta | \mathbf{y}_1 \ldots \mathbf{y}_p \right) &= (k-1) \sum_{i=1}^{p} n_i \ln \left(\tilde{y}_i \right) - \beta^{-1} \sum_{i=1}^{p} n_i \bar{y}_i \\ &- n \left\{ \ln \left[\Gamma \left(\kappa \right) \right] \right\} - n\kappa \ln \left(\beta \right) \end{aligned}$$

Under the null hypothesis the maximum likelihood estimator of the common shape parameter, κ, satisfies

$$\hat{\kappa}\widehat{\beta} = \sum_{i=1}^{p} \nu_i \bar{y}_i \text{ where } \nu_i = \frac{n_i}{n}.$$

So it follows that:

$$\ln \left(\frac{\prod_{i=1}^{p} \tilde{y}^{\nu_i}}{\sum_{i=1}^{p} \nu_i \bar{y}_i} \right) = \ln \left(W \right) = \psi \left(\kappa \right) - \ln \left(\kappa \right).$$

Thus the maximum likelihood estimator satisfies the same equation as in the one-sample problem with the exception that the ratio of the geometric mean of the observations within one sample is replaced with the weighted geometric mean of the p geometric means to the weighted arithmetic means of the individual samples. Thus the statistic, W, treats the values as though they all came from the sample and has the same distribution as that of Bartlett's test statistic for equal variances among p independent and normally distributed populations which have respective sample sizes, $n_1, \ldots, n_p$. Thus the test statistic follows the distribution laid out by Glaser [8], [9], [10], [11], Nandi [18] and Dyer and Keating [6].

5.1.1 Example: Equal Sample Sizes

Consider the failure data in Feiveson and Kulkarni [7] of stress-ruptures of Kevlar-wrapped pressure vessels. These data can also be found in Glaser [12]. Consider the data for two samples obtained from Spool 1 separated by different stress–fractions:

sample 1	
stress fraction	time (hours)
0.791	453.4
	664.5
	930.4
	1,755.5

sample 2	
stress fraction	time (hours)
0.853	444.4
	755.2
	952.2
	1,108.2

These data arise from the typical stress–strain relationship, in which the stress–fraction measures the amount of stress relative to nominal value. Thus higher stress–fractions are indicative of higher loads or stress levels. The time to failure at a specific stress–fraction is then measured in hours. The engineer then wants to determine whether the failure rate has increased at the higher stress–fraction. If the times to failure are modeled with the gamma distribution, then testing the equality of shape parameters (i.e., $H_0 : \kappa_1 = \kappa_2$ vs. $H_1 : \kappa_1 \neq \kappa_2$) becomes a test for equality of failure rates.

At a stress–fraction of 0.791, the geometric, $\tilde{y}_1$, and arithmetic, $\bar{y}_1$, means are 837.5520 and 950.95, respectively. The subsequent ratio, w_1, of geometric to arithmetic mean is 0.8808. At a stress–fraction of 0.791, the geometric, $\tilde{y}_2$, and arithmetic, $\bar{y}_2$, means are 771.4283 and 815.00, respectively. The ratio, w_2, of geometric to arithmetic mean is 0.9465. The geometric and arithmetic means are 803.8105 and 882.9750, respectively when the data are aggregated into one sample of size 8. The ratio is given by 0.9103.

Thus the likelihood ratio test depends upon the:

$$W = \frac{w_1^{\nu_1} w_2^{\nu_2}}{w_0^{\nu_1+\nu_2}} = \frac{w_1^{1/2} w_2^{1/2}}{w_0}$$

Thus the MLE of $\kappa = 5.4842$. The critical value is given by $B_p(\alpha, n) = B_2(0.05, 4) = 0.4780$, where $B_p(\alpha, n)$ is the critical value of Bartlett's test for p populations of equal sample size n. Since $W = 0.9103$ we fail to reject H_0 that the two populations coincide.

5.2 Equality of shape parameters

If one assumes that the shape parameters are equal, then the maximum likelihood estimator of the common shape parameter, κ, satisfies $\hat{\kappa}_i \widehat{\beta}_i = \bar{y}_i, \ \forall\, i = 1, \ldots, p$.

$$\sum_{i=1}^{p} \nu_i \ln(r_i) = \ln\left(\prod_{i=1}^{p} {r_i}^{\nu_i}\right) = \ln(Q) = \psi(\kappa) - \ln(\kappa).$$

where $\nu_i = n_i/n$. Thus the maximum likelihood estimator satisfies the same equation as in the one-sample problem with the exception that the ratio of the geometric mean of the observations within one sample is replaced with the weighted geometric mean of the p geometric means of the individual populations where the weights are the proportions of the individual sample sizes out of the total. Thus a sufficient statistic for inferences about κ is given by

$$Q = \prod_{i=1}^{p} {r_i}^{\nu_i} \Rightarrow \ln(Q) = \sum_{i=1}^{p} \nu_i \ln(r_i).$$

The distribution of each r_i can be expressed as the product of independent beta random variables, (i.e., ${r_i}^{n_i} \sim \prod_{m=1}^{n_i} V_{im}$ where $V_{im} \sim B(\kappa, m/n_i)$). It follows that

$$Q^n = \prod_{i=1}^{p} {r_i}^{n_i} \sim \prod_{i=1}^{p} \prod_{m=1}^{n_i} V_{im} \text{ where}$$

$$V_{im} \sim B(\kappa, m/n_i)\, i = 1, \ldots, p.$$

One can readily see that inferences on κ will be more complex than in the one-sample case. Assuming the shape parameters are equal we can express the distribution of the test-statistic as that of the n^{th} root of a double product of independent beta random variables.

Under the null hypothesis $H_0 : \kappa_1 = \kappa_2 = \cdots = \kappa_p$ vs. $H_1 : \kappa_m \neq \kappa_t$ for some positive integers, m and t, $m \neq t$. For the gamma distribution, the log-likelihood function becomes

$$\begin{aligned} \mathcal{L}_i(\kappa_i, \beta_i|\mathbf{y}_i) &= n_i(\kappa_i - 1)\ln(\tilde{y}_i) - n_i\bar{y}_i\beta_i^{-1} \\ &\quad -n_i\ln[\Gamma(\kappa_i)] - n_i\kappa_i\ln(\beta_i)\,. \end{aligned} \tag{31}$$

5.3 Asymptotics

Under the assumptions of independently sampling n_i observations from the i^{th} population, for each i, $i = 1, \ldots, p$ the distribution of each

$$-2n_i\kappa_i \ln(r_i) \approx \chi^2_{n_i-1}$$

provided $\kappa_i > 1\ \forall\ i, i = 1, \ldots, p$. Therefore, it follows that

$$-2\sum_{i=1}^{p} n_i\kappa_i \ln(r_i) \approx \chi^2_{n-p}.$$

Under the null hypothesis $H_0 : \kappa_1 = \kappa_2 = \cdots = \kappa_p$, since the r_i are independent and by the reproductive property of the chi-square family, we have

$$-2n\kappa \ln(Q) = -2n\kappa \sum_{i=1}^{p} \nu_i \ln(r_i) \approx \chi^2_{n-p}.$$

5.4 Special Case

Without loss of generality let us assume that $n_{\{1\}} = \min\{n_1, n_2, \ldots, n_p\}$.

$$Z_1 = \frac{\left(n_0 - p - n_{\{1\}} + 1\right) n_{\{1\}} \ln\left(r_{\{1\}}\right)}{\left(n_{\{1\}} - 1\right) \sum_{i=2}^{p} n_i \ln(r_i)}.$$

Then it follows that

$$Z_1 \approx F_{n_{\{1\}}-1,\ n_0-p-n_{\{1\}}+1}$$

as the quotient of independent chi-squares divided by their respective degrees of freedom. Let α denote the specified Type I error rate. Our simulations Type I error rates remain in the neighborhood of $\alpha\,(0.05)\ \forall\ n = 2, \ldots, 30$ when $\kappa > 1$.

In special cases, we can simplify this distribution. Suppose that all sample sizes are equal (i.e., $n_1 = \ldots = n_p = n$), define:

$$\begin{aligned}\mathcal{L}_0\left(\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\beta}}|\mathbf{y}_1 \ldots \mathbf{y}_p\right) &= \sum_{i=1}^{p}\left(n_i(\widehat{\kappa}_i - 1)\right)\ln\left(\tilde{y}_i\right) - \sum_{i=1}^{p}\left(n_i\right)\widehat{\kappa}_i \\ &- \sum_{i=1}^{p} n_i\left\{\ln\left[\Gamma\left(\widehat{\kappa}_i\right)\right]\right\} - \sum_{i=1}^{p}\left(n_i\widehat{\kappa}_i\right)\ln\left(\widehat{\beta}_i\right)\end{aligned}$$

5.5 Numerical Methods

We used the secant method to find the maximum likelihood estimators numerically in the gamma distribution. To do this, we must solve for the root(s) of the following function of $\widehat{\kappa}$:

$$\frac{\partial\mathcal{L}(\kappa, \theta|\mathbf{y})}{\partial\kappa} = n\left[\ln\left(Q\right) - \psi(\kappa) + \ln\left(\kappa\right)\right].$$

The secant method requires the continuity of the function as well as upper and lower bounds on the root, $\kappa = \widehat{\kappa}$. The secant method finds a root, c in [a,b], of the function, f(x), based on the Intermediate Value Theorem, where

$$\begin{aligned} f(a) &< 0 \\ f(b) &> 0 \\ f(x) &\text{ is } \text{strictly increasing on [a, b].} \end{aligned}$$

Then there exists a unique root, c in [a,b]. To find the iterative condition, consider the secant-line of the graph of f(x) determined by the points $(a, f(a))$ and $(b, f(b))$ with an equation given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Let c be the x-intercept of this straight line, that is

$$c = a - \frac{b-a}{f(b)-f(a)} f(a).$$

If $f(c) > 0$ then the root is now in [a,c] or if $f(c) < 0$ then the root is in [c,b]. A FORTRAN subroutine is contained in the Appendix that solves for the maximum likelihood equation using the secant method.

Sometimes bounds used in the secant method are far apart and require excessive computation to obtain convergence. A better way to bound the root of κ is based on a table of medians of the distribution of r for various values of n and κ. From the sample data, determine the value of Q. In Table 9 of Keating, Glaser, and Ketchum (1990), find the row of the sample size of the data set on hand. Then move horizontally along the row until r is bracketed above and below. The corresponding values of κ form the lower and upper bounds to $\widehat{\kappa}$.

5.6 FORTRAN Code

FORTRAN Subroutine for finding the MLE's in the Gamma Distribution

```
      dimension x(100) external u common w,s write(6,180)
180 format(' Please enter the sample size: n= ')
      read(5,*) n do 189 i=1,n write(6,181) i
181 format(' Please enter x( ',i2,' ) = ' )
      read(5,*) x(i)
189 continue
      write(10,300) (x(i),i=1,10) write(10,300) (x(i),i=11,20) write(10,300) (x(i),i=21,30)
write(10,300) (x(i),i=31,40) write(10,300) (x(i),i=41,50) write(10,300) (x(i),i=51,60)
write(10,300) (x(i),i=61,70) write(10,300) (x(i),i=71,80) write(10,300) (x(i),i=81,90)
write(10,300) (x(i),i=91,100)
300 format(10f8.2)
      xn=sngl(n) sum1=0.0 sum2=0.0 do 80 i=1,n sum1=sum1+alog(x(i)) sum2=sum2+x(i)
80 continue
```

avg1=sum1/xn avg2=sum2/xn w=exp(avg1)/avg2 s=avg2 write(6,186) w,s write(10,186) w,s

186 format(' the ratio = ',f8.6,' and the mean = ',f12.6)

write(6,184)

184 format(' enter the lower bound on the mle of κ ')

read(5,*) aa write(6,183)

183 format(' enter the upper bound on the mle of κ ')

read(5,*) bb

88 ab=aa-(u(aa)*(bb-aa))/(u(bb)-u(aa)) write(6,*) ab u0=u(ab) if(abs(u0).lt.1.e-6) go to 81 write(6,190) ab,u0 write(10,190) ab,u0

190 format(2f12.6) if(u0.lt.0.0) go to 82 aa=ab go to 88

82 bb=ab go to 88

81 th=s/ab write(6,187) ab,th,ab*th write(10,187) ab,th,ab*th

187 format(/' The mle of κ=',f7.4,', the mle of theta= 1 ',f9.4,' , and the mle of the mean=',f8.4/) write(6,199)

199 format(/' enter the value of x0 '/) read(5,*) x0 c0=x0/th p0=GAMDF(c0,ab) write(6,198) p0,x0 write(10,198) p0,x0

198 format(/' the failure prob = ',f6.4,' at x0 = ', 1 f10.4/) stop end function u(y) common w,s u=alog(y)-psi(y)+alog(w)

65 return end

5.7 Examples of MLE's

Example 1. Consider the following data:

162 200 271 302 393 508 539 629 706 777 884 1,008 1,101 1,182 1,463 1,603 1,984 2,355 and 2,880.

The sample arithmetic mean, $\bar{y} = 997.21051$ divided by the sample geometric mean, $\tilde{y} = 743.70963$, produces a ratio, $Q = 0.74579$. For $n = 19$, the referenced table brackets the root between $a = 1.50$ and $b = 2.00$. Using the Wilk–interpolation scheme developed in Keating, Glaser, and Ketchum (1990), write down $(a, \frac{1}{1-a^*})$, $(c, \frac{1}{1-Q})$, and

$(b, \frac{1}{1-b^*})$ where a^* and b^* are the lower and upper bounds on Q found in Table 9. Then (c, b) are narrower bounds for $\widehat{\kappa}$. Using the Wilk interpolation scheme gives a value of $c = 1.71$. Using these improved bounds: [1.71, 2.00], the solution of $\widehat{\kappa} = 1.8540$ is obtained in 4 iterations of the secant method. If we had chosen more conservatively and set the lower bound to be 1.0 instead of 1.7108, the secant subroutine would have required 15 iterations before convergence.

The corresponding maximum likelihood estimator of the scale parameter, β is given by $\widehat{\beta} = 537.8821$.

Example 2. Consider the following data which are the times between failures of an air conditioning system on Boeing aircraft # 7909.

10, 14, 20, 23, 24, 25, 26, 29, 44, 44, 49, 56, 59, 60, 61, 62, 70, 76, 79, 84, 90, 101, 118, 130, 156, 186, 208, 208, 310.

The sample arithmetic mean, $\bar{y} = 83.51724$ divided by the sample geometric mean, $\tilde{y} = 60.15438$, yields a ratio, $Q = 0.72026$. For $n = 29$, the referenced table brackets the root between $a = 1.50$ and $b = 2.00$. Proceeding as in the preceding example, we obtain $\widehat{\kappa} = 1.671$ and $\widehat{\beta} = 49.9806$.

5.8 Examples of MME's

Example 1 Revisited: Reconsider the following data:

162 200 271 302 393 508 539 629 706 777 884 1,008 1,101 1,182 1,463 1,603 1,984 2,355 and 2,880.

The sample arithmetic mean, $\bar{y} = 997.21051$ and the standard deviation with n as a divisor $S = 740.94837$ yield a method of moments estimator of $\widehat{\kappa}_0 = 1.8113$ $\widehat{\beta} = 550.5402$.

Example 2 Revisited: Reconsider the failures data of an aircraft air conditioning system.

10 14 20 23 24 25 26 29 44 44 49 56 59 60 61 62 70 76 79 84 90 101 118 130 156 186 208 208 310.

The sample arithmetic mean, $\bar{y} = 83.51724$ and the standard deviation with n as a divisor of $S = 69.5744$ yield a method of moments estimators of $\widehat{\kappa} = 1.4410$ and $\widehat{\beta} = 57.9578$. Notice that there is greater disparity between the moment and maximum likelihood estimators in this example.

6 Appendix

As demonstrated earlier, the issue of finding maximum likelihood estimators for a common shape parameter when among several gamma populations reduces to precisely the same problem as in the one population situation. Herein, we produce the derivation for one family for completeness and provide FORTRAN code for the numerical solution.

6.1 The Maximum Likelihood Estimators-One Family

To find the maximum likelihood estimators, $\widehat{\kappa}$ and $\widehat{\beta}$, of the parameters k and β, we solve for the absolute maximum of $\mathcal{L}\left(k, \beta|\mathbf{y}\right)$ using methods of the calculus. The first partial derivative of $\mathcal{L}(k, \beta|\mathbf{y})$ with respect to k is given by

$$\frac{\partial \mathcal{L}(\kappa, \theta|\mathbf{y})}{\partial \kappa} = n\ln\left(\tilde{y}\right) - n\frac{\Gamma'\left(k\right)}{\Gamma\left(k\right)} - n\ln\left(\beta\right).$$

One can simplify this expression by noticing that the digamma function, $\psi(x)$, is defined by

$$\psi\left(x\right) = \frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)}.$$

The first partial derivative of $\mathcal{L}(k, \beta|\mathbf{y})$ with respect to β is given by:

$$\frac{\partial \mathcal{L}(\kappa, \theta|\mathbf{y})}{\partial \theta} = n\bar{y}\beta^{-2} - n\kappa\beta^{-1}.$$

These two equations must be set equal to zero and solved simultaneously. Setting the partial derivative with respect to β equal to zero yields

$$\widehat{\beta} = \frac{\bar{y}}{\widehat{\kappa}}.$$

Using this substitution into the expression for the first partial derivative of the log–likelihood function with respect to κ yields:

$$\frac{\partial \mathcal{L}(\widehat{\kappa}, \widehat{\beta}|\mathbf{y})}{\partial \kappa} = n\left[\ln\left(\frac{\tilde{y}}{\bar{y}}\right) - \psi(\widehat{\kappa}) + \ln(\widehat{\kappa})\right].$$

Notice that this expression involves the components of $\mathbf{y}$ only through the ratio, r, of their geometric to their arithmetic mean, where r_i is formally given as

$$r = \frac{\tilde{y}}{\bar{y}}.$$

This means that the maximum likelihood estimator of κ depends upon the data in the sample only through this ratio. By the Neyman factorization Theorem, we know that r is sufficient for estimation of κ_i. Pitman [20] first proved that r and $\bar{y}$ are independent random variables using properties of invariance. The fact that

$$0 < r < 1$$

with probability 1 is a direct consequence of Jensen's inequality.

$$\frac{\partial \mathcal{L}(\widehat{\kappa}, \widehat{\beta}|\mathbf{y})}{\partial \kappa} = 0 \Rightarrow \ln\left(\frac{\tilde{y}}{\bar{y}}\right) = \psi(\widehat{\kappa}) - \ln(\widehat{\kappa}).$$

Thus we can clearly see that the MLE is a function of the statistic r alone. The distribution of r is quite complex and an efficient series representation of its density has been found only in the last two decades (see Lawless [15]; Keating, Glaser, and Ketchum [14]).

Hence we can solve this equation as a function of $\widehat{\kappa}$. However, to make certain that the solution indeed maximizes the log–likelihood function and hence the likelihood function, we must check that the second partial derivatives with respect to κ and β are negative and that the determinant of the matrix of second partial derivatives is positive.

The entries of the matrix of second partial derivatives is given by

$$\frac{\partial^2 \mathcal{L}(k, \beta|\mathbf{y})}{\partial \kappa^2} = -n\psi'(\kappa) \tag{32}$$

$$\frac{\partial^2 \mathcal{L}(k, \beta|\mathbf{y})}{\partial k \partial \theta} = -n\beta^{-1} \tag{33}$$

$$\frac{\partial^2 \mathcal{L}(k,\beta|\mathbf{y})}{\partial\theta^2} \quad = \quad -2n\bar{y}\beta^{-3} + n\kappa\beta^{-2} \tag{34}$$

where $\psi'(t)$ is the trigamma function. The second partial derivative of the log likelihood function with respect to κ evaluated at $\widehat{\kappa}$ is given by

$$\frac{\partial^2 \mathcal{L}(\widehat{\kappa},\widehat{\beta}|\mathbf{y})}{\partial\kappa^2} = -n\psi'(\widehat{\kappa}) < 0.$$

The inequality follows since the digamma function is a strictly increasing function. Since $\bar{y} = \widehat{\kappa}\widehat{\beta}$, the second partial derivative of the log likelihood function with respect to β evaluated at $(\widehat{\kappa},\widehat{\beta})$ is given by

$$\frac{\partial^2 \mathcal{L}(\widehat{\kappa},\widehat{\beta}|\mathbf{y})}{\partial\theta^2} = -n\widehat{\kappa}\widehat{\beta}^{-2} < 0.$$

The matrix $\mathbf{D}$ of second partial derivatives is given by

$$\mathbf{D}(\kappa,\beta) = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(k,\beta|\mathbf{y})}{\partial\kappa^2} & \frac{\partial^2 \mathcal{L}(k,\beta|\mathbf{y})}{\partial k\partial\theta} \\ \frac{\partial^2 \mathcal{L}(k,\beta|\mathbf{y})}{\partial k\partial\theta} & \frac{\partial^2 \mathcal{L}(k,\beta|\mathbf{y})}{\partial\theta^2} \end{bmatrix}. \tag{35}$$

When the matrix of second partial derivatives is evaluated at the maximum likelihood estimators, we obtain:

$$\mathbf{D}(\widehat{\kappa},\widehat{\beta}) = \begin{bmatrix} -n\psi'(\widehat{\kappa}) & -n\widehat{\beta}^{-1} \\ -n\widehat{\beta}^{-1} & -n\widehat{\kappa}\widehat{\beta}^{-2} \end{bmatrix}. \tag{36}$$

The determinant of the matrix of second partial derivatives evaluated at the maximum likelihood estimators is given by

$$|\mathbf{D}(\widehat{\kappa},\widehat{\beta})| = \frac{n^2}{\widehat{\beta}^2}\left[\widehat{\kappa}\psi'(\widehat{\kappa}) - 1\right].$$

Since

$$\psi'(t) = \frac{\Gamma''(t)\Gamma(t) - [\Gamma'(t)]^2}{[\Gamma(t)]^2} > 0.$$

See Artin [1])). The determinant is positive and therefore the maxima exist.

6.2 FORTRAN Program

```
c This program deletes some print statements which are unnecessary
c (to include in the paper) to simulate the distribution of the log-likelihood
c ratio statistic, ln(Λ), to test equality of shape parameters of two
gamma populations
      USE NUMERICAL LIBRARIES
      implicit real*8(a-h,o-z)
      dimension n(5),x(50),y(50),alpha1(10),alpha2(10),xlnl(50000)
      @,qprob(15), xy(100),q(15),whi(15),wlo(15),simq(10,10,15)
      External WS, u, gamlik
c define quantile probabilites
      qprob(1)=.005d0
      qprob(2)=.010d0
      qprob(3)=.025d0
      qprob(4)=.050d0
      qprob(5)=.100d0
      qprob(6)=.250d0
      qprob(7)=.375d0
      qprob(8)=.500d0
      qprob(9)=.625d0
      qprob(10)=.750d0
      qprob(11)=.900d0
      qprob(12)=.950d0
      qprob(13)=.975d0
      qprob(14)=.990d0
      qprob(15)=.995d0

c generate data from gamma distributions
      eps=0.25d0
      do i=1,10
      alpha1(i)=1.0d0+dfloat(i-1)*.5d0
      alpha2(i)=1.0d0+dfloat(i-1)*.5d0
      write(6,*)'alpha1=', alpha1(i),' alpha2=',alpha2(i)
      write(8,*)'alpha1=', alpha1(i),' alpha2=',alpha2(i)
      end do

      do i=1,5
      n(i)=5+ (i-1)*5
      end do
      nsim=50000

c This loop is to set the parameter alpha
```

```
      do 10 i=1,5
      ii=1+2*(i-1)
      a1=alpha1(ii)
      a2=alpha2(ii)

c this loop is for fixing the sample size


      do 15 j=1,5
c n1=50
c n2=50
      n1=n(j)
      n2=n(j)
      write (6,*)'n1=',n1, ' n2=',n2

c This loop is to generate Nsim values of the likelihood ratio
c statistic to generate its c cut points under H0: Equality of
c shape parameters.
      do 20 k=1,nsim

c Generate n1 values from the gammma distributions with parameters a1 and a2

      call drngam(n1, a1,x)
      call drngam(n2, a2,y)

c compute W1 for sample 1, W2 for sample 2, and and W for the combined sample

c *** define aa and bb for each sample based on method of moment estimates
***
      sumx1=0.0d0
      sumx2=0.0d0
      sumy1=.0d0
      sumy2=0.0d0
      do ii=1, n1
      sumx1=sumx1+x(ii)
      sumx2=sumx2+x(ii)**2
      end do
      do jj=1,n2
      sumy1=sumy1+y(jj)
```

```
      sumy2=sumy2+y(jj)**2
      end do

      xn1=dfloat(n1)
      yn2=dfloat(n2)
      xbar=sumx1/xn1
      ybar=sumy1/yn2
      varx=(xn1*sumx2-sumx1**2)/(xn1*(xn1-1.0))
      vary=(yn2*sumy2-sumy1**2)/(yn2*(yn2-1.0))
      ratx=xbar**2/varx
      raty=ybar**2/vary

      aax=dmax1(ratx - 0.5d0,eps)
      bbx=ratx + 2.d0
      aay=dmax1(raty - 0.5d0,eps)
      bby=raty + 2.0d0

      call WS(x,n1,W1,s1)
      call gamlik(aax,bbx,w1,s1,xk1,th1)
      call WS(y,n2,W2,s2)
      call gamlik(aay,bby,w2,s2,xk2,th2)
```

c compute $\ln(\prod(w_i)^c i)$ and compute k0 and th0

```
      n0=n1+n2
      xn0=dfloat(n0)
      xn1=dfloat(n1)
      xn2=dfloat(n2)
      c1=xn1/xn0
      c2=xn2/xn0
      W0=(W1**c1)*(W2**c2)
```

c Set up initial values for aa bb under H_0
c Set up the grand vector combining x and y

```
      do 40 kk=1,n0
      if (kk .le. n1) go to 35
      xy(kk)=y(kk-n1)
```

```
      go to 40
35 xy(kk)=x(kk)
40 continue

c Find mean and variance of the combined sample
      sum10=0.0d0
      sum20=0.0d0
      do i1=1,n0
      sum10=sum10+xy(i1)
      sum20=sum20+xy(i1)**2
      end do
      xyn0=dfloat(n0)
      xybar=sum10/xyn0
      varxy=(xyn0*sum20-sum10**2)/(xyn0*(xyn0-1.0d0))
      rat0=xybar**2/varxy

      aa0=dmax1(rat0 - 0.5d0,eps)
      bb0=rat0 + 2.0d0

      call WS(xy,n0,W0,s0)
      call gamlik(aa0,bb0,w0,s0,xk0,th0)

c To compute Ln likelihood
      t0=xk0*dlog(w0)
      t1=xn1*xk1*dlog(w1)+xn2*xk2*dlog(w2)
      t2=xn0*xk0*dlog(xk0)-xn1*xk1*dlog(xk1)-xn2*xk2*dlog(xk2)
      t3=xn0*xk0-xn1*xk1-xn2*xk2
      t4=xn0*dlngam(xk0)-xn1*dlngam(xk1)-xn2*dlngam(xk2)
      xlnl(k)=-2.0d0*(xn0*t0-t1+t2-t3-t4)
20 continue

      call deqtil(nsim,xlnl,15,qprob, q,wlo, whi,nmiss)
      write(6,*) 'Percentiles of likelihood ratio statistic:'
      write(6,*)'n1=',n1,' n2=',n2, ' alpha1=',a1, 'alpha2=',a2
      write(8,*)'n1=',n1,' n2=',n2, ' alpha1=',a1, 'alpha2=',a2
      write(6,14)
      write(8,100)
```

```
100 format(2x,'ltprob',' & ',' simquantiles ',' & ', ' simquantiles ',
@' & ', ' low ',8x,'high')
14 format(2x,'ltprob',4x,'simquantiles',2x, 'simquantiles',8x,
@'low',8x,'high')
      do ii=1,15
      simq(i,j,ii)=q(ii)
      write(6,12)qprob(ii), q(ii),simq(i,j,ii), wlo(ii),whi(ii)
      write(8,12)qprob(ii), q(ii),wlo(ii),whi(ii)
12 format(f10.4,2x, 4(f10.4, 2x))
      end do
15 continue
10 continue

c To output tables of quantiles write(8,*) 'Percentiles of -2log(lambda) for equality
@ of Shape parameters'
      do 99 ni=1,5
      write(8,102)
102 format(30x, 'alpha' )
      write(8,101) n(ni), (alpha1(i),i=1,10,2)
101 format(2x, i3,' & ',' quantile ', ' & ', 4(2x,f5.2,' & '),2x,f5.2)
      do 103 jj=1,15
      write(8,104) qprob(jj),(simq(ni,jj,kk),kk=1,5)
104 format('&', f6.3, 5(' & ',f8.3))
103 continue
99 continue
      stop
      end

c Subroutine to compute the mle of k and theta for gamma distribution.
      subroutine gamlik(aa,bb,w,s,xk,th)
      implicit real*8(a-h,o-z)
      external u
88 ab=aa-(u(aa,w)*(bb-aa))/(u(bb,w)-u(aa,w))
      if(ab .lt. 0.0d0) ab=0.25d0
      u0=u(ab,w)
      if(dabs(u0) .lt. 1.0d-6) go to 81
```

```
      if (u0 .lt. 0.0d0) go to 82
      aa=ab
      go to 88
82 bb=ab
      if(dabs((bb-ab)/(aa-ab)).gt.5.d0) go to 88
      aa=(ab+aa)/2.d0
      go to 88
81 th=s/ab
      xk=ab
      return
      end

c Function to compute the likelihood equation for alpha
      function u(y,w)
      implicit real*8(a-h,o-z)
      zz=dpsi(y)
      u=dlog(y)-zz+dlog(w)
      return
      end

c subroutine to compute w=xtilde/xbar and s=xbar
      subroutine WS(x,n,w,s)
      Implicit real*8(a-h,o-z)
      dimension X(100)
      sum1=0.0d0
      sum2=0.0d0
      do i=1,n
      sum1=sum1+dlog(x(i))
      sum2=sum2+x(i)
      end do
      xn=dfloat(n)
      avg1=sum1/xn
      avg2=sum2/xn
      W=dexp(avg1)/avg2
      s=avg2
      return
      end
```

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Table 1: Quantiles for Testing Equality of Shapes for Two Gamma Distributions

n_1	n_2	LTProb	Common Shape Parameter (κ) 1.00	2.00	3.00	4.00	5.00
5	5						
		0.005	0.015	0.011	0.010	0.011	0.011
		0.010	0.029	0.023	0.021	0.023	0.022
		0.025	0.073	0.059	0.055	0.056	0.054
		0.050	0.145	0.121	0.115	0.114	0.108
		0.100	0.295	0.245	0.232	0.228	0.226
		0.250	0.795	0.664	0.642	0.618	0.606
		0.375	1.293	1.092	1.043	0.999	0.985
		0.500	1.887	1.617	1.528	1.475	1.457
		0.625	2.674	2.289	2.164	2.100	2.061
		0.750	3.785	3.228	3.061	2.958	2.922
		0.900	6.296	5.383	5.079	4.928	4.897
		0.950	8.192	6.942	6.565	6.411	6.364
		0.975	10.025	8.526	8.101	7.826	7.857
		0.990	12.483	10.643	10.062	9.798	9.734
		0.995	14.508	12.249	11.504	11.226	11.389

n_1	n_2	LTProb	Common Shape Parameter (κ) 1.00	2.00	3.00	4.00	5.00
10	10						
		0.005	0.013	0.011	0.011	0.010	0.011
		0.010	0.027	0.023	0.021	0.021	0.021
		0.025	0.069	0.058	0.054	0.054	0.055
		0.050	0.138	0.116	0.115	0.110	0.109
		0.100	0.288	0.234	0.236	0.226	0.222
		0.250	0.798	0.652	0.634	0.619	0.609
		0.375	1.292	1.066	1.025	1.007	0.997
		0.500	1.897	1.587	1.513	1.491	1.480
		0.625	2.695	2.256	2.153	2.113	2.084
		0.750	3.810	3.167	3.038	2.973	2.924
		0.900	6.268	5.249	5.057	4.924	4.893
		0.950	8.129	6.837	6.574	6.412	6.363
		0.975	10.005	8.479	8.048	7.900	7.819
		0.990	12.365	10.489	10.038	9.835	9.780
		0.995	14.325	12.091	11.505	11.366	11.224

n_1	n_2	LTProb	Common Shape Parameter (κ) 1.00	2.00	3.00	4.00	5.00
15	15						
		0.005	0.013	0.011	0.011	0.010	0.010
		0.010	0.027	0.021	0.021	0.022	0.020
		0.025	0.070	0.056	0.054	0.055	0.052
		0.050	0.140	0.116	0.113	0.114	0.110
		0.100	0.288	0.240	0.233	0.229	0.232
		0.250	0.798	0.655	0.635	0.617	0.618
		0.375	1.307	1.075	1.039	1.013	1.004
		0.500	1.929	1.595	1.535	1.484	1.475
		0.625	2.698	2.272	2.166	2.099	2.081
		0.750	3.816	3.215	3.055	2.952	2.927
		0.900	6.288	5.342	5.094	4.918	4.858
		0.950	8.202	6.961	6.600	6.411	6.311
		0.975	10.026	8.505	8.167	7.880	7.724
		0.990	12.530	10.759	10.255	9.801	9.627
		0.995	14.592	12.377	11.848	11.287	11.113

n_1	n_2	LTProb	Common Shape Parameter (κ) 1.00	2.00	3.00	4.00	5.00
20	20						
		0.005	0.014	0.010	0.012	0.010	0.011
		0.010	0.029	0.020	0.022	0.022	0.022
		0.025	0.071	0.057	0.055	0.054	0.054
		0.050	0.144	0.118	0.115	0.106	0.107
		0.100	0.295	0.245	0.236	0.224	0.224
		0.250	0.798	0.664	0.633	0.615	0.612
		0.375	1.286	1.094	1.036	1.007	1.001
		0.500	1.888	1.605	1.521	1.488	1.467
		0.625	2.697	2.261	2.170	2.107	2.075
		0.750	3.770	3.219	3.050	2.997	2.933
		0.900	6.255	5.327	5.027	4.999	4.894
		0.950	8.140	6.896	6.573	6.439	6.462
		0.975	9.964	8.548	8.078	7.971	7.857
		0.990	12.404	10.690	10.085	9.957	9.940
		0.995	14.285	12.216	11.647	11.447	11.466

n_1	n_2	LTProb	Common Shape Parameter (κ) 1.00	2.00	3.00	4.00	5.00
25	25						
		0.005	0.014	0.011	0.012	0.012	0.011
		0.010	0.027	0.023	0.024	0.023	0.021
		0.025	0.070	0.057	0.059	0.055	0.053
		0.050	0.143	0.116	0.115	0.107	0.111
		0.100	0.288	0.243	0.233	0.220	0.223
		0.250	0.783	0.662	0.628	0.609	0.603
		0.375	1.273	1.082	1.032	1.010	0.990
		0.500	1.882	1.594	1.518	1.485	1.447
		0.625	2.673	2.259	2.159	2.091	2.056
		0.750	3.781	3.194	3.051	2.963	2.926
		0.900	6.259	5.254	5.085	4.970	4.841
		0.950	8.188	6.799	6.597	6.455	6.278
		0.975	10.055	8.442	8.182	8.000	7.759
		0.990	12.658	10.538	10.155	9.878	9.667
		0.995	14.475	12.322	11.627	11.278	11.098