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Comparison between Constant-stress Testing and Step-stress Testing under Type-I Censoring

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ABSTRACT

By running the life tests at higher stress levels than normal operating conditions, accelerated life testing quickly yields information on the lifetime distribution of a test unit. The lifetime at the design stress is then estimated through extrapolation using a regression model. In constant-stress testing, a unit is tested at a fixed stress level until failure or the termination time point of the test, while step-stress testing allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the test. In this work, the optimal k-level constant-stress and step-stress accelerated life tests are compared for the exponential failure data under complete sampling and Type-I censoring. The objective is to quantify the advantage of using the step-stress testing relative to the constant-stress one. A log-linear relationship between the mean lifetime parameter and stress level is assumed and the cumulative exposure model holds for the effect of changing stress in step-stress testing. The optimal design point is then determined under C-optimality, D-optimality, and A-optimality criteria. The efficiency of step-stress testing compared to constant-stress testing is discussed in terms of the

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ratio of optimal objective functions based on the information matrix.

Keywords: Accelerated life testing; Change-point; Constant-stress testing; Cumulative exposure model; Fisher information; Maximum likelihood estimation; Optimal allocation; Optimal regression design; Step-stress testing; Type-I censoring

JEL Classifications: C13, C16, C24

1 Introduction

With increasing reliability and substantially long life-spans of products, it is often difficult for standard life testing methods under normal operating conditions to obtain sufficient information about the failure time distribution of the products. This difficulty is overcome by accelerated life test (ALT) where the test units are subjected to higher stress levels than normal for rapid failures. By applying more severe stresses, ALT collects information on the parameters of lifetime distributions more quickly. Some key references in the area of ALT are Nelson (1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002). There are two special classes of ALT: constant-stress testing and step-stress testing. In constant-stress testing, a unit is tested at a fixed stress level until failure occurs or the life test is terminated, whichever comes first. On the other hand, step-stress testing allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the test.

The optimal ALT design has attracted great attention in the reliability literature. Miller and Nelson (1983) initiated research in this area by considering a simple step-stress model with exponential failure time distribution under complete sampling. The fundamental model used was the one proposed by Sedyakin (1966), which is known as the *cumulative exposure model*. This model was further discussed and generalized by Bagdonavicius (1978) and Nelson (1980). Bai, Kim and Lee (1989) extended the results of Miller and Nelson (1983) to the time-censored situation while Khamis and Higgins (1996) studied the case of three stress levels. Khamis and Higgins (1998) also considered the problem under Weibull lifetime distribution for units subjected to stress. Khamis (1997) then compared constant-stress ALT and step-stress ALT for Weibull failure data. Yeo and Tang (1999) investigated the optimality problem in the situation when a target acceleration factor was pre-specified. Recently, exact conditional inference for a step-stress model with exponential competing risks was studied by Balakrishnan and Han (2008), Han and Balakrishnan (2010). Ng, Balakrishnan and Chan (2007) discussed the problem of determining the optimal sample size allocation for a general k-level model with extreme value regression while Gouno, Sen and Balakrishnan (2004), Balakrishnan and Han (2009) discussed the problem of determining the optimal stress duration under progressive Type-I censoring; see also Han *et al.* (2006) for some related comments.

The main focus of this article is to investigate the advantage of using step-stress ALT relative to constant-stress ALT. Assuming a log-linear relationship between the mean lifetime parameter and stress level, with the cumulative exposure model for the effect of changing stress in step-stress ALT, the optimal design point is determined under various optimality criteria. In particular, the cases of complete sampling and Type-I censoring are considered under exponential lifetime distribution for units subjected to stress. Using the ratio of optimal objective functions as a measure of relative efficiency, comparison of k-level step-stress testing to k-level constant-stress testing is discussed through a numerical study.

The rest of the paper is organized as follows. Section 2 describes the model under study, derives the MLEs of the model parameters and the associated Fisher information for k-level constant-stress ALT and step-stress ALT. Section 3 then defines the three optimality criteria based on the Fisher information (*viz.*, variance, determinant, and trace) and talks about the existence of optimal design points in each case under complete sampling and Type-I censoring. For the purpose of further comparison between constant-stress and step-stress tests, the expected total time spent on test is also obtained in Section 4. Finally, Section 5 provides the results of a numerical study and discusses the relative efficiency of these two classes of ALT under consideration.

2 Model description, MLEs and Fisher information

Let us first define $x_1 < x_2 < \ldots < x_k$ to be the ordered stress levels to be used in the test. Then, the following assumptions form the basis of constructing both the constant-stress model and the step-stress model.

Assumptions 2.1.

(i) For any stress level x_i, the lifetime of a test unit follows an exponential distribution. That is, the probability density function (PDF) and the corresponding cumulative distribution function (CDF) of a test unit at stress level x_i are

$$f_i(t) = \frac{1}{\theta_i} \exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{2.1}$$

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right), \qquad 0 < t < \infty, \tag{2.2}$$

respectively;

(ii) At stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log \theta_i = \alpha + \beta x_i, \tag{2.3}$$

where the regression parameters α and β are unknown and need to be estimated.

No notational distinction is made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$.

2.1 k-level constant-stress test under Type-I censoring

A constant-stress test under Type-I censoring proceeds as follows. For $i = 1, 2, \dots, k$, $N_i \equiv n\pi_i$ units are allocated on test at stress level x_i such that $\sum_{i=1}^{k} N_i = n$ or equivalently, $\sum_{i=1}^{k} \pi_i = 1$. $\pi_i = N_i/n$ is the allocation proportion of units (out of total n units under the test) assigned to stress level x_i. The allocated units are then tested until time τ_i at which point all the surviving items are withdrawn, thereby terminating the life test. Let n_i denote the number of units failed at stress level x_i in time interval $[0, \tau_i)$ and $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \dots, n_i$ while $N_i - n_i$ denotes the number of units censored at time τ_i. Obviously, when there is no right censoring (*viz.*, $\tau_i = \infty$ and $n_i = N_i$), this situation corresponds to the k-level constant-stress testing under complete sampling as a special case.

Now, under the assumption (i), the joint probability density function (JPDF) of $\mathbf{n} = (n_1, n_2, \dots, n_k)$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \dots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \dots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}, \mathbf{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right]\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{2.4}$$

where

$$U_i = \sum_{l=1}^{n_i} y_{i,l} + (N_i - n_i)\tau_i, \qquad i = 1, 2, \dots, k. \tag{2.5}$$

Note that U_i in (2.5) is the *Total Time on Test* statistic at stress level x_i. Now, using (2.4) and the assumption (ii), the log-likelihood function of (α, β) can be written as

$$l(\alpha, \beta) = -\alpha \sum_{i=1}^{k} n_i - \beta \sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i \exp\left[-(\alpha + \beta x_i)\right]. \tag{2.6}$$

Upon differentiating (2.6) with respect to α and β, the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained as simultaneous solutions to the following two equations:

$$\left[\sum_{i=1}^{k} n_i\right]\left[\sum_{i=1}^{k} U_i x_i \exp(-\hat{\beta} x_i)\right] = \sum_{i=1}^{k} n_i x_i \sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i), \tag{2.7}$$

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i)}{\sum_{i=1}^{k} n_i}\right). \tag{2.8}$$

As shown above, $\hat{\alpha}$ and $\hat{\beta}$ are non-linear functions of random quantities and thus, statistical inference with these MLEs can be based on the asymptotic distributional result that the vector $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as a bivariate normal with mean vector (α, β) and variance-covariance matrix $\mathbf{I}_n^{-1}(\alpha, \beta)$, where $\mathbf{I}_n(\alpha, \beta)$ is the Fisher information matrix. By using the following properties of the counts and order statistics, $\mathbf{I}_n(\alpha, \beta)$ is obtained.

Properties 2.1.

(1) For $i = 1, 2, \ldots, k$, the random variable n_i has a binomial distribution with parameters $(N_i, F_i(\tau_i))$.

(2) For $i = 1, 2, \ldots, k$, given n_i, the random variables $y_{i,l}$, $l = 1, 2, \ldots, n_i$, are distributed jointly as order statistics from a random sample of size n_i from a right-truncated exponential distribution with PDF $f_{i,\tau_i}(z) = \dfrac{f_i(z)}{F_i(\tau_i)}$ for $0 \leq z \leq \tau_i$.

Theorem 2.1. *Under this setup of the constant-stress test with Type-I censoring, the Fisher information matrix is*

$$\mathbf{I}_n(\alpha, \beta) = n \begin{pmatrix} \sum_{i=1}^{k} A_i & \sum_{i=1}^{k} A_i x_i \\ \sum_{i=1}^{k} A_i x_i & \sum_{i=1}^{k} A_i x_i^2 \end{pmatrix}, \tag{2.9}$$

where

$$A_i = \pi_i F_i(\tau_i), \qquad i = 1, 2, \ldots, k. \tag{2.10}$$

2.2 k-level step-stress test under Type-I censoring

For $i = 1, 2, \ldots, k$, let us first define n_i to be the number of units failed at stress level x_i in time interval $[\tau_{i-1}, \tau_i)$ and $y_{i,l}$ to be the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$. Furthermore, let N_i denote the number of units operating and remaining on test at the start of stress level x_i $\left(viz., N_i = n - \sum_{j=1}^{i-1} n_j\right)$. Then, a step-stress test under Type-I censoring proceeds as follows.

A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time τ_1 at which point the stress is changed to x_2. The test is continued on remaining $N_2 = n - n_1$ units until time τ_2 at which the stress is changed to x_3, and so on. Finally, at time τ_k, all the surviving items are withdrawn, thereby terminating the life test. Note that the number of surviving items at time τ_k is $n - \sum_{i=1}^{k} n_i = N_k - n_k$. Obviously, when there is no right censoring (*viz.*, $\tau_k = \infty$ and $n_k = N_k$), this situation corresponds to the k-level step-stress testing under complete sampling as a special case.

Now, under the cumulative exposure model along with the assumption (i), the PDF and CDF of a test unit are

$$f(t) = \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] f_i(t - \tau_{i-1}) \qquad \text{if} \begin{cases} \tau_{i-1} \le t \le \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases}, \tag{2.11}$$

$$F(t) = 1 - \left[\prod_{j=1}^{i-1} S_j(\Delta_j)\right] S_i(t - \tau_{i-1}) \qquad \text{if} \begin{cases} \tau_{i-1} \le t \le \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases}, \tag{2.12}$$

where $\Delta_j = \tau_j - \tau_{j-1}$ is the step duration at stress level x_j, and $f_i(t)$ and $F_i(t)$ are as given in (2.1) and (2.2), respectively. Then, using (2.11) and (2.12), the JPDF of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as in (2.4) where

$$U_i = \sum_{l=1}^{n_i} (y_{i,l} - \tau_{i-1}) + (N_i - n_i)\Delta_i, \qquad i = 1, 2, \ldots, k. \tag{2.13}$$

Again, note that U_i in (2.13) is the *Total Time on Test* statistic at stress level x_i. Using (2.4) and the assumption (ii), the log-likelihood function of (α, β) can be written as in (2.6) and as a result, we obtain the MLEs $\hat{\alpha}$ and $\hat{\beta}$ as simultaneous solutions to (2.7) and (2.8) with U_i given in (2.13).

Just like in the case of constant-stress testing, $\hat{\alpha}$ and $\hat{\beta}$ are non-linear functions of random quantities and hence, inference using these MLEs are based on the asymptotic distributional result $\Big($*viz.*, $(\hat{\alpha}, \hat{\beta})' \stackrel{.}{\sim} BVN\big((\alpha, \beta)',\ \mathbf{I}_n^{-1}(\alpha, \beta)\big)\Big)$. Again, by using the following properties of the counts and order statistics (which include Properties 2.1 as a special case), we can derive the expression of $\mathbf{I}_n(\alpha, \beta)$ as well as the expectation of N_i.

Properties 2.2.

(1) The random variable n_1 has a binomial distribution with parameters $(n, F_1(\Delta_1))$. For $i = 2, 3, \ldots, k$, given $n_1, n_2, \ldots, n_{i-1}$, the random variable n_i has a binomial distribution with parameters $(N_i, F_i(\Delta_i))$.

(2) Given $n_1, n_2, \ldots, n_i$, the random variables $(y_{i,l} - \tau_{i-1})$, $l = 1, 2, \ldots, n_i$, are distributed jointly as order statistics from a random sample of size n_i from a right-truncated exponential distribution with PDF $f_{i,\Delta_i}(z) = \dfrac{f_i(z)}{F_i(\Delta_i)}$ for $0 \leq z \leq \Delta_i$ and $i = 1, 2, \ldots, k$.

Lemma 2.1. *For $i = 1, 2, \ldots, k$,*

$$E[N_i] = n \prod_{j=1}^{i-1} S_j(\Delta_j). \tag{2.14}$$

Theorem 2.2. *Under this setup of the step-stress test with Type-I censoring, the Fisher information matrix is as in (2.9) where*

$$A_i = \left[\prod_{j=1}^{i-1} S_j(\Delta_j) \right] F_i(\Delta_i), \qquad i = 1, 2, \ldots, k. \tag{2.15}$$

3 Optimality criteria and existence of optimal design points

In this section, we define different optimality criteria for determining the optimal design points, which then can be used to compare between the multi-level constant-stress test and step-stress test. For the k-level constant-stress testing, the focus is to determine the optimal allocation proportions $\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*, \ldots, \pi_k^*)$ with $\pi_k^* = 1 - \sum_{i=1}^{k-1} \pi_i^*$ while it is to determine the optimal stress durations $\boldsymbol{\Delta}^* = (\Delta_1^*, \Delta_2^*, \ldots, \Delta_k^*)$ for the k-level step-stress testing. These objective functions are purely based on the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ derived in the preceding section. Unlike A_i defined in

Gouno, Sen and Balakrishnan (2004), A_i's in (2.10) and (2.15) are always positive, and this, in turn, eliminates any disconcerting anomalies and ensures a positive determinant of $\mathbf{I}_n(\alpha, \beta)$ as well as a positive variance function. Therefore, there is no particular restriction on the search region for the optimal design points in these cases.

3.1 C-optimality

In an ALT experiment, researchers often wish to estimate the parameters of interest with maximum precision and minimum variability possible. In both the constant-stress and step-stress settings under consideration here, such a parameter of interest is the mean lifetime of a unit at the use-condition (*viz.*, θ_0). For this purpose, we consider an objective function given by

$$\begin{aligned} \phi(\cdot) &= n \ \mathrm{AVar}\big(\log \hat{\theta}_0\big) = n \ \mathrm{AVar}\big(\hat{\alpha} + \hat{\beta} x_0\big) \\ &= n \ (1 \ x_0) \mathbf{I}_n^{-1}(\alpha, \beta) \binom{1}{x_0} \\ &= \frac{2 \displaystyle\sum_{i=1}^{k} A_i (x_i - x_0)^2}{\displaystyle\sum_{i=1}^{k} \sum_{j=1}^{k} A_i A_j (x_i - x_j)^2}, \end{aligned} \tag{3.1}$$

where AVar stands for asymptotic variance and x_0 is the normal use-stress (*i.e.*, $x_0 < x_1$). The C-optimal design points are the ones that minimize $\phi(\cdot)$ in (3.1). In the case of $k = 2$ (*i.e.*, the case of a simple stress testing), the objective function in (3.1) reduces to

$$\begin{aligned} \phi(\cdot) &= \frac{A_1(x_1 - x_0)^2 + A_2(x_2 - x_0)^2}{A_1 A_2 (x_2 - x_1)^2} \\ &= \frac{(1 + \xi_0)^2}{A_1} + \frac{\xi_0^2}{A_2}, \end{aligned} \tag{3.2}$$

where $\xi_0 = \dfrac{x_1 - x_0}{x_2 - x_1}$.

Theorem 3.1. *In the case of a simple constant-stress test under Type-I censoring, the C-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1 + \rho} \qquad \text{and} \qquad \pi_2^* = \frac{\rho}{1 + \rho},$$

where $\rho^2 = \dfrac{\xi_0^2}{(1+\xi_0)^2}\dfrac{F_1(\tau_1)}{F_2(\tau_2)} = \left(\dfrac{x_1 - x_0}{x_2 - x_0}\right)^2 \dfrac{F_1(\tau_1)}{F_2(\tau_2)}$.

Corollary 3.1. *In the case of a simple constant-stress test under Type-I censoring, the C-optimality allocates an equal number of test units at each stress level (viz., $\pi_1^* = \pi_2^* = 1/2$) when $(x_1 - x_0)^2 F_1(\tau_1) = (x_2 - x_0)^2 F_2(\tau_2)$.*

Theorem 3.2. *In the case of a k-level constant-stress test under complete sampling (i.e., no right censoring by letting $\tau_i \to \infty$ for $i = 1, 2, \ldots, k$), the C-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1+\rho}, \qquad \pi_i^* = 0 \quad \textit{for} \quad i = 2, 3, \ldots, k-1, \qquad \textit{and} \qquad \pi_k^* = \frac{\rho}{1+\rho},$$

where $\rho = \dfrac{x_1 - x_0}{x_k - x_0}$.

Corollary 3.2. *In the case of a simple constant-stress test under complete sampling, the C-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1+\rho} \qquad \textit{and} \qquad \pi_2^* = \frac{\rho}{1+\rho},$$

where $\rho = \dfrac{\xi_0}{1+\xi_0} = \dfrac{x_1 - x_0}{x_2 - x_0}$.

Theorem 3.3. *In the case of a simple step-stress test under Type-I censoring with an equal step duration (viz., $\Delta_1 = \Delta_2 = \Delta$), there exists a C-optimal step duration Δ^* which is the unique solution to the equation $\phi'(\Delta) = 0$.*

Theorem 3.4. *In the case of a simple step-stress test under complete sampling (i.e., no right censoring by letting $\tau_2 \to \infty$), the C-optimal stress change point is*

$$\Delta_1^* = \theta_1 \log\left(1 + \frac{1}{\rho}\right),$$

where $\rho = \dfrac{\xi_0}{1+\xi_0} = \dfrac{x_1 - x_0}{x_2 - x_0}$.

3.2 D-optimality

Another optimality criterion often used in planning ALT is based on the determinant of the Fisher information matrix, which equals to the reciprocal of the determinant of the asymptotic variance-covariance matrix. Note that the overall volume of the Wald-type joint confidence region of (α, β) is proportional to $|\mathbf{I}_n^{-1}(\alpha, \beta)|^{1/2}$ at a fixed level of confidence. In other words, it is inversely proportional to $|\mathbf{I}_n(\alpha, \beta)|^{1/2}$, the square root of the determinant of $\mathbf{I}_n(\alpha, \beta)$. Consequently, a larger value of $|\mathbf{I}_n(\alpha, \beta)|$ would correspond to a smaller asymptotic joint confidence ellipsoid of (α, β) and thus a higher joint precision of the estimators of α and β. Motivated by this, our second objective function is simply given by

$$\begin{aligned} \delta(\cdot) &= n^{-2}|\mathbf{I}_n(\alpha, \beta)| \\ &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} A_i A_j (x_i - x_j)^2. \end{aligned} \tag{3.3}$$

The D-optimal design points are obtained by maximizing (3.3) for the maximal joint precision of $\left(\hat{\alpha}, \hat{\beta}\right)$. For $k = 2$, the objective function (3.3) reduces to

$$\delta(\cdot) = A_1 A_2 (x_2 - x_1)^2. \tag{3.4}$$

Theorem 3.5. *In the case of a simple constant-stress test under Type-I censoring, the D-optimality allocates an equal number of test units at each stress level (viz., $\pi_1^* = \pi_2^* = 1/2$).*

Theorem 3.6. *In the case of a k-level constant-stress test under complete sampling (i.e., no right censoring by letting $\tau_i \to \infty$ for $i = 1, 2, \ldots, k$), the D-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{2}, \qquad \pi_i^* = 0 \quad \text{for} \quad i = 2, 3, \ldots, k-1, \qquad \text{and} \qquad \pi_k^* = \frac{1}{2}.$$

Corollary 3.3. *In the case of a simple constant-stress test under complete sampling, the D-optimality allocates an equal number of test units at each stress level (viz., $\pi_1^* = \pi_2^* = 1/2$).*

Theorem 3.7. *In the case of a simple step-stress test under Type-I censoring with an equal step duration (viz.,* $\Delta_1 = \Delta_2 = \Delta$*), there exists a D-optimal step duration* Δ^* *which is the unique solution to the equation* $\theta_1 F_1(\Delta) S_2(\Delta) = \theta_2 [1 - 2S_1(\Delta)] F_2(\Delta)$*.*

Theorem 3.8. *In the case of a simple step-stress test under complete sampling (i.e., no right censoring by letting* $\tau_2 \to \infty$*), the D-optimal stress change point is the median of the distribution at stress level* x_1 *(viz.,* $\Delta_1^* = \theta_1 \log 2$*).*

3.3 A-optimality

Another optimality criterion considered in our study is based on the trace of the first-order approximation of the variance-covariance matrix of the MLEs. It is identical to the sum of the diagonal elements of $\mathbf{I}_n^{-1}(\alpha, \beta)$. The A-optimality criterion provides an overall measure of the average variance of the parameter estimates and gives the sum of the eigenvalues of the inverse of the Fisher information matrix. The A-optimal design points minimize the objective function defined by

$$\begin{aligned} a(\cdot) &= n\ tr\big(\mathbf{I}_n^{-1}(\alpha,\beta)\big) \\ &= \frac{2 \sum_{i=1}^{k} A_i (1 + x_i^2)}{\sum_{i=1}^{k} \sum_{j=1}^{k} A_i A_j (x_i - x_j)^2}. \end{aligned} \tag{3.5}$$

In the case of the simple stress testing ($k = 2$), the objective function in (3.5) becomes

$$\begin{aligned} a(\cdot) &= \frac{A_1(1 + x_1^2) + A_2(1 + x_2^2)}{A_1 A_2 (x_2 - x_1)^2} \\ &= \frac{\xi_2^2}{A_1} + \frac{\xi_1^2}{A_2}, \end{aligned} \tag{3.6}$$

where $\xi_i = \dfrac{\sqrt{1 + x_i^2}}{x_2 - x_1}$ for $i = 1, 2$.

Theorem 3.9. *In the case of a simple constant-stress test under Type-I censoring, the A-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1+\rho} \qquad and \qquad \pi_2^* = \frac{\rho}{1+\rho},$$

where $\rho^2 = \dfrac{\xi_1^2}{\xi_2^2}\dfrac{F_1(\tau_1)}{F_2(\tau_2)} = \left(\dfrac{1+x_1^2}{1+x_2^2}\right)\dfrac{F_1(\tau_1)}{F_2(\tau_2)}$.

Corollary 3.4. *In the case of a simple constant-stress test under Type-I censoring, the A-optimality allocates an equal number of test units at each stress level (viz.,* $\pi_1^* = \pi_2^* = 1/2$*) when* $(1+x_1^2)F_1(\tau_1) = (1+x_2^2)F_2(\tau_2)$*.*

Theorem 3.10. *In the case of a k-level constant-stress test under complete sampling (i.e., no right censoring by letting* $\tau_i \to \infty$ *for* $i = 1, 2, \ldots, k$*), the A-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1+\rho}, \qquad \pi_i^* = 0 \quad \textit{for} \quad i = 2, 3, \ldots, k-1, \qquad \textit{and} \qquad \pi_k^* = \frac{\rho}{1+\rho},$$

where $\rho^2 = \dfrac{1+x_1^2}{1+x_k^2}$.

Corollary 3.5. *In the case of a simple constant-stress test under complete sampling, the A-optimal allocation proportions are*

$$\pi_1^* = \frac{1}{1+\rho} \qquad \textit{and} \qquad \pi_2^* = \frac{\rho}{1+\rho},$$

where $\rho^2 = \dfrac{\xi_1^2}{\xi_2^2} = \dfrac{1+x_1^2}{1+x_2^2}$.

Theorem 3.11. *In the case of a simple step-stress test under Type-I censoring with an equal step duration (viz.,* $\Delta_1 = \Delta_2 = \Delta$*), there exists an A-optimal step duration* Δ^* *which is the unique solution to the equation* $a'(\Delta) = 0$*.*

Theorem 3.12. *In the case of a simple step-stress test under complete sampling (i.e., no right censoring by letting* $\tau_2 \to \infty$*), the A-optimal stress change point is*

$$\Delta_1^* = \theta_1 \log\left(1 + \frac{1}{\rho}\right),$$

where $\rho^2 = \dfrac{\xi_1^2}{\xi_2^2} = \dfrac{1+x_1^2}{1+x_2^2}$.

Remark 3.1. *It is of interest to note that for a simple constant-stress test, the D-optimal design allocates an equal number of test units at each stress level regardless of the stress levels used, the presence of Type-I censoring nor the time points of censoring at any stress level.*

Remark 3.2. *From the results above, we see that in planning a k-level constant-stress test under complete sampling, there is no need to assign the test units at any stress level other than the lowest* (x_1) *and the highest* (x_k) *levels. If the D-optimality is concerned, one should allocate 50% of the units to stress level* x_1 *and the remaining 50% to stress level* x_k*. If the C-optimality is considered, then one should allocate* $100\pi_1^*\%$ *of the units to stress level* x_1 *and the remaining* $100(1-\pi_1^*)\%$ *to stress level* x_k *where* π_1^* *is defined accordingly. Although using only two extreme stress levels does not allow us to check the linearity assumption in the model, the methodology developed here can be used to design a different optimal allocation plan subject to a particular allocation constraint; see for example Ng, Balakrishnan and Chan (2007).*

Remark 3.3. *For a multi-level constant-stress test under complete sampling, it is observed that under the C-optimality and A-optimality criteria,* $0 < \rho < 1$ *since* $x_0 < x_1 < x_2$*, and therefore* $\pi_1^* = \dfrac{1}{1+\rho} > \dfrac{1}{2}$*. This means that the C-optimal and A-optimal designs allocate more test units at the lower stress level than does the D-optimal design.*

Remark 3.4. *For a simple step-stress test under complete sampling, it is observed that under the C-optimality and A-optimality criteria,* $0 < \rho < 1$ *since* $x_0 < x_1 < x_2$*, and therefore* $F_1(\Delta_1^*) = \dfrac{1}{1+\rho} > \dfrac{1}{2}$*. This means that the C-optimal and A-optimal stress change time points are larger than the D-optimal point, which is the median of the lifetime distribution at stress level* x_1*.*

Remark 3.5. *For a simple step-stress test under complete sampling, it is noted that for each optimality considered here, as* θ_1 *increases,* Δ_1^* *increases in a manner such that the ratio of* Δ_1^* *to* θ_1 *stays constant across the values of* θ_1*. This is because the optimal stress change point is a fixed percentile from the distribution at stress level* x_1*, irrespective of the MTTF at that level. From the numerical study of Balakrishnan and Han (2009), the same feature also prevailed in the case of a k-level step-stress test under Type-I censoring as well as under progressive Type-I censoring.*

All the optimality criteria considered here, as well as some other information-based criteria, have been used extensively in the design selection process for linearly designed experiments. From a practitioner's point of view, the choice of the optimality criterion will be certainly guided by the objective of the experiment. In cases where the planner is more interested in the precise estimation of the MTTF θ_0 at normal use-condition, the C-optimality is surely the criterion of choice. On the other hand, if one is more concerned about estimating the mean function given in assumption (ii) or estimating the regression parameters α and β with high precision, a more reasonable criterion of choice should be the D-optimality or A-optimality.

4 Expected total time on test

In order to compare the total time spent on test between the constant-stress test and the step-stress test, the expected total time spent on test is computed. In general, for a life test under Type-I censoring with the sample size n and the censoring time point τ, the expected total time spent on test is

$$T^{\epsilon} = E[Y_{n:n}|Y_{n:n} < \tau]Pr(Y_{n:n} < \tau) + \tau \; Pr(Y_{n:n} \geq \tau), \tag{4.1}$$

where $Y_{n:n}$ is the largest order statistic from a lifetime distribution characterized by, say, the PDF $f(t)$ and the CDF $F(t)$. Note that given $Y_{n:n} < \tau$, $Y_{n:n}$ is distributed as the largest order statistic from a random sample of size n from a right-truncated distribution with PDF $f_{\tau}(t) = \dfrac{f(t)}{F(\tau)}$ for $0 \leq t \leq \tau$. Using this property, the conditional expectation above is expressed as

$$\begin{aligned} E[Y_{n:n}|Y_{n:n} < \tau] &= n \int_0^{\tau} y \left[\frac{F(y)}{F(\tau)}\right]^{n-1} \left[\frac{f(y)}{F(\tau)}\right] dy \\ &= \frac{n}{[F(\tau)]^n} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \int_0^{\tau} y[S(y)]^l f(y) \; dy. \end{aligned} \tag{4.2}$$

Since $Pr(Y_{n:n} < \tau) = [F(\tau)]^n$, we now have

$$T^{\epsilon} = \sum_{l=1}^{n} \binom{n}{l} (-1)^{l+1} l \int_0^{\tau} y[S(y)]^{l-1} f(y) \; dy + \tau\big(1 - [F(\tau)]^n\big). \tag{4.3}$$

4.1 k-level constant-stress test under Type-I censoring

From (4.3), the expected time spent at stress level x_i is

$$\begin{aligned} T_i^\epsilon &= \theta_i \sum_{l=1}^{N_i} \binom{N_i}{l} \frac{(-1)^{l+1}}{l} \left(1 - [S_i(\tau_i)]^l\right) \\ &= \theta_i \sum_{l=1}^{N_i} \binom{N_i}{l} \frac{(-1)^{l+1}}{l} F_i(l\tau_i) \end{aligned} \tag{4.4}$$

for $i = 1, 2, \ldots, k$. When there is only one facility available for constant-stress testing, the life tests at different stress levels have to proceed in a sequential way. Then, the expected total time spent on test is simply the sum of the expected time spent at each stress level.

Theorem 4.1. *In the case of a k-level constant-stress test under Type-I censoring, the expected total time spent on test is*

$$T^\epsilon = \sum_{i=1}^{k} T_i^\epsilon = \sum_{i=1}^{k} \theta_i \sum_{l=1}^{N_i} \binom{N_i}{l} \frac{(-1)^{l+1}}{l} F_i(l\tau_i).$$

Corollary 4.1. *In the case of a simple constant-stress test under Type-I censoring, the expected total time spent on test is*

$$T^\epsilon = \theta_1 \sum_{l=1}^{N_1} \binom{N_1}{l} \frac{(-1)^{l+1}}{l} F_1(l\tau_1) + \theta_2 \sum_{l=1}^{N_2} \binom{N_2}{l} \frac{(-1)^{l+1}}{l} F_2(l\tau_2).$$

Theorem 4.2. *In the case of a k-level constant-stress test under complete sampling (i.e., no right censoring by letting $\tau_i \to \infty$ for $i = 1, 2, \ldots, k$), the expected total time spent on test is*

$$T^\epsilon = \sum_{i=1}^{k} \theta_i \sum_{l=1}^{N_i} \binom{N_i}{l} \frac{(-1)^{l+1}}{l}.$$

Corollary 4.2. *In the case of a simple constant-stress test under complete sampling, the expected total time spent on test is*

$$T^\epsilon = \theta_1 \sum_{l=1}^{N_1} \binom{N_1}{l} \frac{(-1)^{l+1}}{l} + \theta_2 \sum_{l=1}^{N_2} \binom{N_2}{l} \frac{(-1)^{l+1}}{l}.$$

Remark 4.1. *When there are k multiple facilities available for constant-stress testing, the life tests at different stress levels may proceed in a partially or completely parallel way. If the test proceeds in* *a completely parallel way with the same starting time point, the total time spent on the entire test is determined by the longest time taken among tests at different stress levels. For a k-level constant-stress test under complete sampling, the expected total time spent on test is*

$$T^{\epsilon} = \int_0^{\infty} t \; f_{par}(t) \; dt,$$

where

$$\begin{aligned} F_{par}(t) &= \prod_{i=1}^{k} [F_i(t)]^{N_i}, & 0 < t < \infty, \\ f_{par}(t) &= \frac{d}{dt} F_{par}(t) = \prod_{i=1}^{k} [F_i(t)]^{N_i} \sum_{i=1}^{k} N_i \frac{f_i(t)}{F_i(t)}, & 0 < t < \infty. \end{aligned}$$

For a simple constant-stress test under complete sampling, this is

$$T^{\epsilon} = \sum_{i=1}^{2} \theta_i \sum_{l=1}^{N_i} \sum_{l'=0}^{N_{i'}} \binom{N_i}{l} \binom{N_{i'}}{l'} \frac{(-1)^{l+l'+1}}{l} \left(1 + \frac{l'\theta_i}{l\theta_{i'}}\right)^{-2},$$

where $i' = 1, 2$ and $i' \neq i$.

4.2 k-level step-stress test under Type-I censoring

Using (4.3) again, the expected total time spent on a step-stress test is derived.

Theorem 4.3. *In the case of a k-level step-stress test under Type-I censoring (or under complete sampling), the expected total time spent on test is*

$$\begin{aligned} T^{\epsilon} &= \sum_{l=1}^{n} \binom{n}{l} (-1)^{l+1} l \sum_{i=1}^{k} \int_{\tau_{i-1}}^{\tau_i} \frac{y}{\theta_i} \left[\prod_{j=1}^{i-1} S_j(\Delta_j) \right]^l [S_i(y - \tau_{i-1})]^l \; dy + \tau_k \big(1 - [F(\tau_k)]^n\big) \\ &= \sum_{i=1}^{k} \theta_i \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} [S(\tau_{i-1})]^l \big(1 - [S_i(\Delta_i)]^l\big) \\ &= \sum_{i=1}^{k} \theta_i \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \left[\prod_{j=1}^{i-1} S_j(\Delta_j) \right]^l F_i(l\Delta_i). \end{aligned}$$

Corollary 4.3. *In the case of a simple step-stress test under Type-I censoring, the expected total time spent on test is*

$$T^{\epsilon} = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \Big(\theta_1 F_1(l\Delta_1) + \theta_2 S_1(l\Delta_1) F_2(l\Delta_2)\Big).$$

Corollary 4.4. *In the case of a simple step-stress test under complete sampling, the expected total time spent on test is*

$$T^{\epsilon} = \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+1}}{l} \Big(\theta_1 F_1(l\Delta_1) + \theta_2 S_1(l\Delta_1) \Big).$$

5 Numerical results

A numerical study was conducted in order to investigate the relative efficiency of step-stress testing compared to constant-stress testing and to evaluate it as a function of varying parameters (*i.e.*, the sample size, MTTF, and the number of stress levels). For the purpose of illustration, we considered equi-spaced stress levels as $x_i = x_0 + id$ with the use-stress level $x_0 = 10$ and the stress increment $d = 5$. Under this setup, optimizing with respect to any optimality criterion under considerations is independent of the values of x_0 and d. We also chose the ordered MTTF as

$$\theta_{i+1} = \rho\theta_i, \qquad i = 1, 2, \ldots, k-1, \qquad 0 < \rho < 1,$$

with selected choices of θ_1 and ρ. Hence, a decreasing geometric sequence of MTTF was simulated with an increasing arithmetic sequence of stress levels.

For a k-level step-stress test under Type-I censoring with an equal step duration Δ, Table 1 presents the values of the optimal step duration Δ_C^*, Δ_D^* and Δ_A^*, which are independent of the sample size n. Rather than the specific values of the optimal stress durations, the table is intended to provide a qualitative insight into the way the optimal choice changes as a function of the relevant parameters. Table 2 presents the corresponding optima of each objective function described in Section 3. These optima are independent of the values of θ_1 as well as the sample size n. From Table 1, it is observed that $\Delta_C^* > \Delta_A^* > \Delta_D^*$ except for the simple step-stress case with $\rho = 0.5$. This order, however, is a consequence of the specific setting chosen and does not necessarily hold for general stress levels. Also, for a given k and ρ, the ratios Δ_C^*/Δ_D^* and Δ_D^*/Δ_A^* remain constant over varying ranges of θ_1. The dependence of the optimal values on ρ is less noticeable for smaller values of k. The behavior of the optimal Δ as a function of the MTTF values is also interesting. For fixed k and ρ, as θ_1 increases, Δ_C^*, Δ_D^* and Δ_A^* increase in a manner such that the ratios Δ_C^*/θ_1, Δ_D^*/θ_1 and Δ_A^*/θ_1 are constant across the values of θ_1. This translates to Δ_C^*, Δ_D^* and Δ_A^* being fixed percentiles from the stage-1 distribution, irrespective of the value of θ_1. The same observation was made in Balakrishnan and Han (2009) under progressive Type-I censoring.

Using the optimal step durations obtained in Table 1 as the censoring time points at each stress level, the allocation proportions $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_k)$ were then optimized for a k-level constant-stress test under Type-I censoring. Table 3 presents the values of these optimal allocation proportions $\boldsymbol{\pi}_C^*$, $\boldsymbol{\pi}_D^*$ and $\boldsymbol{\pi}_A^*$, and Table 4 presents the corresponding optima of each objective function described in Section 3. These optimal allocation proportions and the corresponding optima are independent of the values of θ_1 as well as the sample size n. From Table 3, it is observed that $\pi_{1,C}^* > \pi_{1,A}^* > \pi_{1,D}^*$ regardless of the values of k and ρ. It is also interesting to note that except for the first and last stress levels, the C-optimality and the D-optimality do not allocate any test units in the intermediate stress levels. The same observation was analytically proven under complete sampling as stated in Theorems 3.2 and 3.10. The D-optimality on the other hand allocates an equal number of test units at two stress levels with no units allocated in any other stress levels.

In order to compare the total time taken between the constant-stress test and the step-stress test under the optimal conditions, the expected total time spent on each test was computed based on the results derived in Section 4 with the optimal step durations in Table 1 and the optimal allocation proportions in Table 3. The results are given in Tables 5 and 6, respectively. It is observed that in either testing schemes alone, the expected total test time does not change much as k increases. However, the step-stress testing takes longer than the constant-stress testing as k increases although they are quite similar when $k = 2$. In both cases, it takes longer to complete the test as the sample size n increases. From Tables 5 and 6, it is also observed that $T_C^\epsilon > T_A^\epsilon > T_D^\epsilon$ regardless of the values of other parameters. As observed similarly in Table 1, for fixed k and ρ, as θ_1 increases, T_C^ϵ, T_D^ϵ and

Table 1: Optimal step durations for a step-stress test under Type-I censoring

		$k=2$			$k=3$			$k=4$		
		Δ_C^*	Δ_D^*	Δ_A^*	Δ_C^*	Δ_D^*	Δ_A^*	Δ_C^*	Δ_D^*	Δ_A^*
$\theta_1 = 100$	$\rho = 0.1$	109.87	69.78	84.73	109.82	8.18	84.16	109.82	8.08	84.16
	$\rho = 0.3$	111.91	82.85	88.94	56.47	27.09	34.29	54.44	9.26	15.35
	$\rho = 0.5$	114.83	94.06	93.18	72.10	45.03	51.05	50.35	23.13	29.89
$\theta_1 = 300$	$\rho = 0.1$	329.60	209.35	254.18	329.00	24.53	252.47	329.00	24.24	252.47
	$\rho = 0.3$	335.73	248.56	266.82	169.42	81.27	102.87	163.33	27.79	46.04
	$\rho = 0.5$	344.48	282.18	279.55	216.31	135.08	153.15	151.06	69.39	89.67
$\theta_1 = 500$	$\rho = 0.1$	549.33	348.92	423.64	549.00	40.88	421.00	550.00	40.40	421.00
	$\rho = 0.3$	559.56	414.27	444.70	282.37	135.45	171.45	272.21	46.32	76.74
	$\rho = 0.5$	574.14	470.31	465.91	360.51	225.13	255.25	251.77	115.65	149.44

Table 2: Corresponding optima of the objective functions

for a step-stress test under Type-I censoring

	$k=2$			$k=3$			$k=4$		
	ϕ_{ss}^*	δ_{ss}^*	a_{ss}^*	ϕ_{ss}^*	δ_{ss}^*	a_{ss}^*	ϕ_{ss}^*	δ_{ss}^*	a_{ss}^*
$\rho = 0.1$	9.00	6.24	49.17	9.00	9.44	49.13	9.00	9.45	49.13
$\rho = 0.3$	9.08	5.76	50.42	7.87	13.00	31.57	7.85	19.65	28.76
$\rho = 0.5$	9.36	5.04	53.63	6.55	13.32	27.50	6.04	23.88	20.55

Table 3: Optimal allocation proportions for a constant-stress test under Type-I censoring

		$\boldsymbol{\pi}_C^*$	$\boldsymbol{\pi}_D^*$	$\boldsymbol{\pi}_A^*$
$k = 2$	$\rho = 0.1$	(0.710, 0.290)	(0.500, 0.500)	(0.638, 0.362)
	$\rho = 0.3$	(0.707, 0.293)	(0.500, 0.500)	(0.628, 0.372)
	$\rho = 0.5$	(0.697, 0.303)	(0.500, 0.500)	(0.611, 0.389)
$k = 3$	$\rho = 0.1$	(0.786, 0, 0.214)	(0, 0.500, 0.500)	(0.688, 0, 0.312)
	$\rho = 0.3$	(0.820, 0, 0.180)	(0.500, 0, 0.500)	(0.753, 0, 0.247)
	$\rho = 0.5$	(0.803, 0, 0.197)	(0.500, 0, 0.500)	(0.711, 0, 0.289)
$k = 4$	$\rho = 0.1$	(0.830, 0, 0, 0.170)	(0, 0.500, 0, 0.500)	(0.726, 0, 0, 0.274)
	$\rho = 0.3$	(0.861, 0, 0, 0.139)	(0, 0.500, 0, 0.500)	(0.841, 0, 0, 0.159)
	$\rho = 0.5$	(0.863, 0, 0, 0.137)	(0.500, 0, 0, 0.500)	(0.789, 0, 0, 0.211)

Table 4: Corresponding optima of the objective functions

for a constant-stress test under Type-I censoring

	$k = 2$			$k = 3$			$k = 4$		
	ϕ_{cs}^*	δ_{cs}^*	a_{cs}^*	ϕ_{cs}^*	δ_{cs}^*	a_{cs}^*	ϕ_{cs}^*	δ_{cs}^*	a_{cs}^*
$\rho = 0.1$	11.90	3.14	68.97	5.46	3.49	23.24	3.87	13.86	13.36
$\rho = 0.3$	11.90	3.30	68.98	7.75	5.64	38.00	5.72	6.43	39.81
$\rho = 0.5$	12.07	3.23	70.81	6.80	7.57	31.01	6.03	9.79	24.88

Table 5: Expected total time on test for the optimal step-stress test under Type-I censoring

			$k=2$			$k=3$			$k=4$		
			T_C^ϵ	T_D^ϵ	T_A^ϵ	T_C^ϵ	T_D^ϵ	T_A^ϵ	T_C^ϵ	T_D^ϵ	T_A^ϵ
$n=5$	$\theta_1=100$	$\rho=0.1$	118.54	85.19	98.04	118.50	17.60	97.56	118.50	17.43	97.56
		$\rho=0.3$	143.01	121.74	126.64	99.56	62.52	73.14	97.46	29.36	42.49
		$\rho=0.5$	161.74	146.26	145.50	136.30	106.22	114.27	115.14	73.77	86.54
	$\theta_1=300$	$\rho=0.1$	355.62	255.56	294.13	355.15	52.80	292.68	355.15	52.28	292.68
		$\rho=0.3$	429.03	365.22	379.93	298.68	187.56	219.42	292.38	88.09	127.47
		$\rho=0.5$	485.23	438.78	436.51	408.91	318.67	342.81	345.43	221.30	259.61
	$\theta_1=500$	$\rho=0.1$	592.70	425.94	490.22	592.44	87.99	487.98	593.23	87.13	487.98
		$\rho=0.3$	715.06	608.70	633.21	497.80	312.59	365.70	487.31	146.81	212.45
		$\rho=0.5$	808.71	731.29	727.51	681.51	531.11	571.35	575.72	368.83	432.68
$n=10$	$\theta_1=100$	$\rho=0.1$	127.92	92.04	105.50	127.88	18.37	104.99	127.88	18.20	104.99
		$\rho=0.3$	163.68	138.06	144.02	112.26	68.56	80.72	109.71	31.54	46.23
		$\rho=0.5$	188.80	167.28	166.23	158.29	119.50	129.87	132.12	81.73	97.25
	$\theta_1=300$	$\rho=0.1$	383.77	276.12	316.50	383.24	55.12	314.96	383.24	54.59	314.96
		$\rho=0.3$	491.04	414.17	432.06	336.79	205.68	242.15	329.13	94.61	138.69
		$\rho=0.5$	566.41	501.83	498.69	474.86	358.50	389.61	396.36	245.20	291.74
	$\theta_1=500$	$\rho=0.1$	639.61	460.21	527.50	639.32	91.87	525.13	640.20	90.98	525.13
		$\rho=0.3$	818.40	690.28	720.10	561.32	342.80	403.59	548.55	157.68	231.14
		$\rho=0.5$	944.02	836.39	831.15	791.44	597.50	649.35	660.60	408.67	486.23
$n=20$	$\theta_1=100$	$\rho=0.1$	134.85	98.69	112.21	134.81	19.05	111.70	134.81	18.87	111.70
		$\rho=0.3$	181.74	151.33	158.65	122.02	73.44	86.69	119.03	33.24	49.04
		$\rho=0.5$	209.67	180.89	179.51	176.33	128.52	141.38	146.28	87.34	105.57
	$\theta_1=300$	$\rho=0.1$	404.56	296.07	336.64	404.02	57.14	335.10	404.02	56.60	335.10
		$\rho=0.3$	545.21	453.98	475.96	366.05	220.31	260.06	357.08	99.71	147.11
		$\rho=0.5$	629.00	542.67	538.53	529.00	385.57	424.13	438.85	262.01	316.70
	$\theta_1=500$	$\rho=0.1$	674.27	493.45	561.07	673.97	95.23	558.70	674.87	94.34	558.70
		$\rho=0.3$	908.69	756.63	793.26	610.09	367.19	433.44	595.13	166.19	245.18
		$\rho=0.5$	1048.33	904.45	897.56	881.67	642.62	706.88	731.42	436.69	527.84

Table 6: Expected total time on test for the optimal constant-stress test under Type-I censoring

			$k=2$			$k=3$			$k=4$		
			T_C^ϵ	T_D^ϵ	T_A^ϵ	T_C^ϵ	T_D^ϵ	T_A^ϵ	T_C^ϵ	T_D^ϵ	T_A^ϵ
$n=5$	$\theta_1=100$	$\rho=0.1$	113.71	77.83	94.68	104.67	8.64	80.72	103.77	7.23	81.94
		$\rho=0.3$	134.62	113.47	125.02	64.99	39.17	43.04	56.74	12.91	18.04
		$\rho=0.5$	167.40	139.88	146.50	94.43	72.41	72.50	62.34	37.76	41.22
	$\theta_1=300$	$\rho=0.1$	341.12	233.50	284.05	313.64	25.93	242.16	310.94	21.69	245.82
		$\rho=0.3$	403.86	340.40	375.06	194.96	117.51	129.11	170.22	38.72	54.11
		$\rho=0.5$	502.19	419.65	439.51	283.30	217.23	217.49	187.02	113.27	123.65
	$\theta_1=500$	$\rho=0.1$	568.54	389.17	473.42	523.27	43.22	403.79	519.58	36.14	409.90
		$\rho=0.3$	673.10	567.33	625.11	324.93	195.85	215.18	283.69	64.53	90.18
		$\rho=0.5$	836.99	699.41	732.51	472.17	362.05	362.49	311.70	188.79	206.08
$n=10$	$\theta_1=100$	$\rho=0.1$	126.97	92.10	104.98	110.58	10.36	85.71	109.23	8.22	84.06
		$\rho=0.3$	163.43	141.41	144.75	69.93	45.52	47.39	57.14	15.00	19.38
		$\rho=0.5$	190.60	173.55	168.76	106.79	84.47	87.75	62.63	43.24	46.40
	$\theta_1=300$	$\rho=0.1$	380.92	276.29	314.93	331.31	31.08	257.12	327.26	24.65	252.17
		$\rho=0.3$	490.29	424.22	434.24	209.79	136.57	142.18	171.42	45.01	58.14
		$\rho=0.5$	571.81	520.64	506.29	320.38	253.41	263.25	187.89	129.72	139.21
	$\theta_1=500$	$\rho=0.1$	634.87	460.48	524.88	552.83	51.79	428.76	547.04	41.08	420.51
		$\rho=0.3$	817.15	707.03	723.74	349.65	227.62	236.96	285.70	75.02	96.90
		$\rho=0.5$	953.02	867.73	843.82	533.97	422.35	438.76	313.15	216.20	232.02
$n=20$	$\theta_1=100$	$\rho=0.1$	134.32	98.97	110.64	111.88	11.10	86.60	109.99	8.37	84.38
		$\rho=0.3$	181.17	154.15	156.68	75.16	49.47	53.86	59.39	16.36	20.27
		$\rho=0.5$	210.56	184.50	180.82	118.82	88.64	95.65	72.61	45.45	51.66
	$\theta_1=300$	$\rho=0.1$	402.97	296.92	331.91	335.20	33.30	259.81	329.52	25.11	253.15
		$\rho=0.3$	543.52	462.45	470.03	225.47	148.40	161.59	178.18	49.08	60.81
		$\rho=0.5$	631.68	553.49	542.47	356.47	265.92	286.96	217.83	136.35	154.97
	$\theta_1=500$	$\rho=0.1$	671.62	494.86	553.18	559.34	55.49	433.23	550.86	41.85	422.13
		$\rho=0.3$	905.86	770.75	783.38	375.78	247.33	269.32	296.96	81.80	101.35
		$\rho=0.5$	1052.80	922.49	904.11	594.12	443.20	478.27	363.04	227.25	258.28

Table 7: Efficiency of the step-stress testing to the constant-stress testing under Type-I censoring

	$k=2$			$k=3$			$k=4$		
Optimality	C	D	A	C	D	A	C	D	A
$\rho=0.1$	1.32	1.99	1.40	0.61	2.70	0.47	0.43	0.68	0.27
$\rho=0.3$	1.31	1.75	1.37	0.99	2.30	1.20	0.73	3.06	1.38
$\rho=0.5$	1.29	1.56	1.32	1.04	1.76	1.13	1.00	2.44	1.21

T_A^ϵ increase in a manner such that the ratios T_C^ϵ/θ_1, T_D^ϵ/θ_1 and T_A^ϵ/θ_1 are constant across the values of θ_1, irrespective of the value of n.

To formally assess the efficiency of the step-stress testing compared to the constant-stress testing under the optimal condition discussed in this section, pairwise ratios of the optima under each criterion were computed using the results obtained in Tables 2 and 4. The values are tabulated in Table 7 where the number greater than 1 indicates higher efficiency of the step-stress test compared to the constant-stress one. As expected, these ratios are invariant across the values of θ_1 and the sample size n. It is also observed from Table 7 that the highest efficiency is achieved by the D-optimality, followed by the A-optimality, and then by the C-optimality in general. In most cases, the step-stress testing is proven to be more superior compared to the corresponding constant-stress testing even under the optimal situations.

References

Bagdonavicius, V., 1978. Testing the hypothesis of additive accumulation of damages. Probab. Theory Application 23, 403–408.

Bagdonavicius, V., Nikulin, M., 2002. Accelerated Life Models: Modeling and Statistical Analysis. Chapman & Hall, Boca Raton, FL.

Bai, D.S., Kim, M.S., Lee, S.H., 1989. Optimum simple step-stress accelerated life tests with censoring. IEEE

Trans. Reliability 38, 528–532.

Balakrishnan, N., Han, D., 2008. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. J. Statist. Plann. Inference 138, 4172–4186.

Balakrishnan, N., Han, D., 2009. Optimal step-stress testing for progressively Type-I censored data from exponential distribution. J. Statist. Plann. Inference 139, 1782–1798.

Gouno, E., Sen, A., Balakrishnan, N., 2004. Optimal step-stress test under progressive Type-I censoring. IEEE Trans. Reliability 53, 383–393.

Han, D., Balakrishnan, N., 2010. Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. Computational Statist. Data Analysis 54, 2066–2081.

Han, D., Balakrishnan, N., Sen, A., Gouno, E., 2006. Corrections on Optimal step-stress test under progressive Type-I censoring. IEEE Trans. Reliability 55, 613–614.

Khamis, I.H., 1997. Comparison between constant and step-stress tests for Weibull models. International J. Quality and Reliability Management 14, 74–81.

Khamis, I.H., Higgins, J.J., 1996. Optimum 3-step step-stress tests. IEEE Trans. Reliability 45, 341–345.

Khamis, I.H., Higgins, J.J., 1998. A new model for step-stress testing. IEEE Trans. Reliability 47, 131–134.

Meeker, W.Q., Escobar, L.A., 1998. Statistical Methods for Reliability Data. Wiley, New York.

Miller, R., Nelson, W., 1983. Optimum simple step-stress plans for accelerated life testing. IEEE Trans. Reliability 32, 59–65.

Nelson, W., 1980. Accelerated life testing - step-stress models and data analysis. IEEE Trans. Reliability 29, 103–108.

Nelson, W., 1990. Accelerated Testing: Statistical Models, Test Plans, and Data Analyses. Wiley, New York.

Ng, H.K.T., Balakrishnan. N., Chan, P.S., 2007. Optimal sample size allocation for tests with multiple levels of stress with extreme value regression. Naval Research Logistics 54, 237–249.

Sedyakin, N.M., 1966. On one physical principle in reliability theory (in Russian). Techn. Cybernetics 3, 80–87.

Yeo, K.P., Tang, L.C., 1999. Planning step-stress life-test with a target acceleration-factor. IEEE Trans. Reliability 48, 61–67.