THE UNIVERSITY OF TEXAS AT SAN ANTONIO, COLLEGE OF BUSINESS

Working Paper SERIES

Date May 1, 2013

WP # 0026MSS-694-2013

Inference for a step-stress model with competing risks from the GE distribution under Type-I censoring

David Han
Department of Mangement Science and Statistics
University of Texas at San Antonio

Debasis Kundu
Department of Mathematics and Statistics
Indian Institute of Technology at Kanpur India

UTSA COLLEGE OF BUSINESS KNOWLEDGE FOR A NEW WORLD

ONE UTSA CIRCLE
SAN ANTONIO, TEXAS 78249-0631
210 458-4317 | BUSINESS.UTSA.EDU

Inference for a step-stress model with competing risks from the GE distribution under Type-I censoring

David Han[1*] **and Debasis Kundu**[2]

[1] *Department of Management Science and Statistics, University of Texas at San Antonio, Texas, USA 78249*

[2] *Department of Mathematics and Statistics, Indian Institute of Technology at Kanpur, India 208016*

ABSTRACT

In reliability analysis, accelerated life-testing allows gradual increment of stress levels on test units during an experiment. In a special class of accelerated life tests known as step-stress tests, the stress levels increase discretely at pre-fixed time points, allowing the experimenter to obtain information on the lifetime parameters more quickly than under normal operating conditions. Moreover, when a test unit fails, there are often more than one fatal cause for the failure, such as mechanical or electrical. In this article, we consider the step-stress model under Type-I censoring when the lifetime distributions of the different risk factors are independent generalized exponential. Under this setup, we derive the maximum likelihood estimates of the unknown scale and shape parameters of the different causes with the assumption of cumulative damage. Using the asymptotic distributions and the parametric bootstrap method, we discuss the construction of confidence intervals for the parameters. The precision of the estimates and the performance of the confidence intervals are also assessed through extensive Monte Carlo simulations, and finally, the methods of inference discussed here is illustrated with an

*Corresponding author: *david.han@utsa.edu* – The author would like to thank the support from the College of Business research grant program.

example.

Keywords: Accelerated life-testing; Competing risks; Confidence interval; Cumulative damage model; Generalized exponential distribution; Maximum likelihood estimation; Step-stress model; Type-I censoring

JEL Classifications: C13, C16, C24

1 Introduction

In order to guarantee the service life and performance of a product, or even to compare alternative manufacturing designs, life-testing under normal operating conditions is obviously most desirable. However, due to the continual improvement in manufacturing design and technology, one often experiences difficulty in obtaining sufficient information about the failure time distribution of the products. As the products become highly reliable with substantially long life-spans, time-consuming and expensive tests are often required to collect enough failure data, which are necessary to draw inference about the relationship of lifetime with the external stress variables. In such situations, the standard life-testing methods are not appropriate, especially when developing prototypes of new products. This difficulty is overcome by accelerated life tests (ALT) wherein the units are subjected to higher stress levels in order to cause rapid failures. ALT allows the experimenter to apply more severe stresses to obtain information on the parameters of the lifetime distributions more quickly than would be possible under normal operating conditions. Some key references in the field of ALT include Nelson and Meeker (1978), Nelson (1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

A special class of the accelerated life-testing, known as step-stress testing, allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the experiment for maximal flexibility and adjustability. This model has attracted great attention in the reliability literature. Sedyakin (1966) proposed one of the fundamental models in this area, known as the *cumulative damage* or *cumulative exposure* model. This model has been further discussed and generalized by Bagdonavicius (1978) and Nelson (1980). Recently, exact conditional inference for a step-stress model with exponential competing risks was developed by Balakrishnan and Han (2008), Han and Balakrishnan (2010). Gouno, Sen and Balakrishnan (2004), Balakrishnan and Han (2009) discussed the problem of determining the optimal stress duration under progressive Type-I censoring; see also Han *et al.* (2006) for some related comments. More recently, Han and Ng (2013) quantified the advantage of using the step-stress ALT relative to the constant-stress ALT under several optimality criteria in the situations of complete sampling and Type-I censoring. For a concise review of step-stress models, readers are referred to Gouno and Balakrishnan (2001) and Balakrishnan (2009).

Furthermore, in reliability analysis, it is common that a failure is associated with one of several fatal risk factors the test unit is exposed to. Since it is not usually possible to study the test units with an isolated risk factor, it becomes necessary to assess each risk factor in the presence of other risk factors. In order to analyze such a competing risks model, each failure observation must come in a bivariate form composed of a failure time and the cause of failure. Cox (1959), David and Moeschberger (1978), Klein and Basu (1981, 1982), and Crowder (2001) have all investigated the competing risks models and considered some specific parametric lifetime distributions for each risk factor. In this paper, we consider the case when the lifetime distribution of each risk factor is two-parameter generalized exponential (GE).

In this paper, we consider the problem of point and interval estimations for a general step-stress model under Type-I censoring when the lifetime distributions of the different risk factors are independent GE. The rest of the paper is organized as follows. Using the cumulative damage model for the effect of changing stress in step-stress ALT, Section 2 describes the model under study and derives the MLEs of the scale and shape parameters of different risk factors. Based on the asymptotic distributions of the MLEs, we construct the confidence intervals for the unknown parameters as well as the confidence intervals by a parametric bootstrap method in Section 3. In Section 4, the precision of the estimates and the performance of the confidence intervals are investigated in terms of bias, mean squared error (MSE), and probability coverage via extensive Monte Carlo simulations. In Section 5, we present a numerical example to illustrate the methods of inference developed in this article, and Section 6 is devoted to some concluding remarks and future works in this direction.

2 Model Description and MLEs

Let us first define (x_0 <) $x_1 < x_2 < \ldots < x_k$ to be the k (≥ 2) ordered stress levels and $0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \infty$ to be the pre-fixed stress change time points being used in the step-stress ALT. A random sample of n identical units is placed on the test under the initial stress level x_1 (or x_0 for a partially accelerated life test, PALT). The successive failure times are then recorded along with the information about which risk factor caused each failure. At the first pre-fixed time τ_1, the stress level is increased to x_2 and the life test continues until the next pre-fixed time τ_2 at which the stress level is increased to x_3. The life test continues in this fashion until the pre-specified censoring time τ_k. When all n units fail before τ_k or when τ_k is unbounded (*viz.*, $\tau_k \to \infty$), then a complete set of failure observations would result for this step-stress test (*viz.*, no censoring). Suppose each unit fails by one of r (≥ 2) fatal risk factors and the time-to-failure by each competing risk has an independent GE distribution which obeys the cumulative damage model. With a constant shape parameter $\alpha_j > 0$ for the risk factor j across the stress levels being used, let $\lambda_{ij} > 0$ be the scale parameter for the risk factor j at the stress level x_i for $1 \leq i \leq k$ and $1 \leq j \leq r$. Then, the cumulative distribution function (CDF) of the lifetime T_j due to the risk factor j is given by

$$G_j(t) = G_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) = \left[1 - \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)\right]^{\alpha_j}$$
$$\text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{2.1}$$

for $1 \leq j \leq r$ where $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$, and $\Delta_i = \tau_i - \tau_{i-1}$ is the step duration at the stress level x_i. The corresponding probability density function (PDF) of T_j is given by

$$g_j(t) = g_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) = \alpha_j \lambda_{ij} \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)$$
$$\times \left[1 - \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)\right]^{\alpha_j - 1}$$
$$\text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{2.2}$$

for $1 \leq j \leq r$. Since only the smallest of $T_1, T_2, \ldots, T_r$ is observed, let $T = \min\{T_1, T_2, \ldots, T_r\}$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$F(t) = F(t; \boldsymbol{\lambda}, \boldsymbol{\alpha}) = 1 - S(t) = 1 - \prod_{j=1}^{r} \big(1 - G_j(t)\big)$$
$$= 1 - \prod_{j=1}^{r} \left\{1 - \left[1 - \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)\right]^{\alpha_j}\right\}$$
$$\text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{2.3}$$

$$f(t) = f(t; \boldsymbol{\lambda}, \boldsymbol{\alpha}) = \left[\sum_{j=1}^{r} h_j(t)\right]\left[\prod_{j=1}^{r} \big(1 - G_j(t)\big)\right] = \left[\sum_{j=1}^{r} h_j(t)\right] S(t)$$
$$= \left[\sum_{j=1}^{r} h_j(t)\right] \prod_{j=1}^{r} \left\{1 - \left[1 - \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)\right]^{\alpha_j}\right\}$$
$$\text{if} \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{2.4}$$

respectively, where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\star 1}, \boldsymbol{\lambda}_{\star 2}, \ldots, \boldsymbol{\lambda}_{\star r})$ with $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, and $h_j(t)$ is the hazard rate function of the risk factor j defined by

$$\begin{aligned} h_j(t) &= h_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) = \frac{g_j(t)}{1 - G_j(t)} \\ &= \frac{\alpha_j \lambda_{ij} \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\Big]^{\alpha_j - 1}}{1 - \Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\Big]^{\alpha_j}} \\ &\qquad \text{if } \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \end{aligned} \tag{2.5}$$

for $1 \le j \le r$. Furthermore, let C denote the indicator for the cause of failure. Then, the joint PDF of (T, C) is given by

$$\begin{aligned} f_{T,C}(t, j) &= g_j(t) \prod_{\substack{j'=1 \\ j' \ne j}}^{r} \big(1 - G_{j'}(t)\big) = h_j(t)\ S(t) \\ &= \alpha_j \lambda_{ij} \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\Big]^{\alpha_j - 1} \\ &\qquad \times \prod_{\substack{j'=1 \\ j' \ne j}}^{r} \Big\{1 - \Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj'}\Delta_l - \lambda_{ij'}(t - \tau_{i-1})\Big)\Big]^{\alpha_{j'}}\Big\} \\ &\qquad \text{if } \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \end{aligned} \tag{2.6}$$

for $t > 0$ and $1 \le j \le r$. Based on these, the relative risk imposed on a test unit at the stress level x_i due to the risk factor j is given by

$$\begin{aligned} \pi_{ij} = P\big[C = j \big| \tau_{i-1} < T < \tau_i\big] &= [S(\tau_{i-1}) - S(\tau_i)]^{-1} \int_{\tau_{i-1}}^{\tau_i} h_j(t) S(t)\ dt \\ &= E\big[h_j^{\pi}(T) \big| \tau_{i-1} < T < \tau_i\big] \end{aligned}$$

for $1 \le i \le k$ and $1 \le j \le r$ where $h_j^{\pi}(t)$ is the hazard proportion of the risk factor j at time $t > 0$ defined by

$$h_j^{\pi}(t) = h_j(t) \Big/ \sum_{j'=1}^{r} h_{j'}(t), \qquad t > 0.$$

Hence, the relative risks are simply the expected proportions of each hazard rate for the corresponding risk factor in the given time frame of stress level.

With the life-testing scheme described above, the following ordered failure times will be observed:

$$\Big\{\tau_{i-1} < t_{i;1} < t_{i;2} < \cdots < t_{i;n_{i\oplus}} < \tau_i\Big\}$$

for $i = 1, 2, \ldots, k$ where $n_{i\oplus}$ denotes the (observed) total number of units failed at the stress level x_i (*i.e.*, in time interval $[\tau_{i-1}, \tau_i)$) and $t_{i;l}$ denotes the l-th ordered failure time of $n_{i\oplus}$ units at the stress level x_i, $l = 1, 2, \ldots, n_{i\oplus}$. Let n_{ij} denote the (observed) number of units failed at the stress level x_i due to the risk factor j and let $n_{\oplus j}$ denote the (observed) total number of units failed by the risk factor j. Also, let $n_{\oplus\oplus}$ ($\leq n$) denote the (observed) accumulated number of failures until the censoring time τ_k according to the testing scheme such that $n_{i\oplus} = \displaystyle\sum_{j=1}^{r} n_{ij}$, $n_{\oplus j} = \displaystyle\sum_{i=1}^{k} n_{ij}$, and $n_{\oplus\oplus} = \displaystyle\sum_{i=1}^{k} n_{i\oplus} = \displaystyle\sum_{j=1}^{r} n_{\oplus j} = \displaystyle\sum_{i=1}^{k}\sum_{j=1}^{r} n_{ij}$. Since each failure time is also accompanied by the corresponding cause of failure, let $\boldsymbol{c} = (c_{1;1}, c_{1;2}, \ldots, c_{k;n_{k\oplus}})$ be the observed sequence of the cause of failure corresponding to the observed failure times $\boldsymbol{t} = (t_{1;1}, t_{1;2}, \ldots, t_{k;n_{k\oplus}})$. Whenever appropriate, no notational distinction will be made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$. Using (2.1)–(2.6), the likelihood function of $\boldsymbol{\theta} = (\boldsymbol{\lambda}, \boldsymbol{\alpha})$ based on this Type-I censored data is then formulated as

$$L(\boldsymbol{\theta}) = L\big(\boldsymbol{\theta}|(\boldsymbol{t}, \boldsymbol{c})\big) = \frac{n!}{(n - n_{\oplus\oplus})!}\left\{\prod_{i=1}^{k}\prod_{l=1}^{n_{i\oplus}} f_{T,C}\left(t_{i;l}, c_{i;l}\right)\right\}\Big\{1 - F(\tau_k)\Big\}^{n - n_{\oplus\oplus}} \tag{2.7}$$

and the corresponding log-likelihood function of $\boldsymbol{\theta}$ is obtained from (2.7) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= l\big(\boldsymbol{\theta}|(\boldsymbol{t}, \boldsymbol{c})\big) = \log L(\boldsymbol{\theta}) \\ &= \left\{\sum_{i=1}^{k}\sum_{l=1}^{n_{i\oplus}} \log g_{c_{i;l}}(t_{i;l})\right\} + \left\{\sum_{i=1}^{k}\sum_{l=1}^{n_{i\oplus}}\sum_{\substack{j=1 \\ j\neq c_{i;l}}}^{r} \log\big(1 - G_j(t_{i;l})\big)\right\} \\ &\quad + (n - n_{\oplus\oplus})\left\{\sum_{j=1}^{r} \log\big(1 - G_j(\tau_k)\big)\right\}. \end{aligned} \tag{2.8}$$

After differentiating $l(\boldsymbol{\theta})$ in (2.8) with respect to λ_{ij} and α_j, we obtain the likelihood equations as

$$\begin{aligned}
0 = \frac{\partial}{\partial \lambda_{ij}} l(\boldsymbol{\theta}) \quad &= \quad \frac{n_{ij}}{\lambda_{ij}} - U_{ij} + \sum_{i'=1}^{k} \sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1) \frac{1 - \left[G_j(t_{i';l})\right]^{1/\alpha_j}}{\left[G_j(t_{i';l})\right]^{1/\alpha_j}} \Big[\Delta_i \delta(i' > i) \\
&\quad + (t_{i;l} - \tau_{i-1})\delta(i' = i)\Big] \delta(c_{i';l} = j) - \sum_{i'=1}^{k} \sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})}{\lambda_{i'j}[1 - G_j(t_{i';l})]} \Big[\Delta_i \delta(i' > i) \\
&\quad + (t_{i;l} - \tau_{i-1})\delta(i' = i)\Big] \delta(c_{i';l} \neq j) - (n - n_{\oplus\oplus}) \frac{g_j(\tau_k)}{\lambda_{kj}[1 - G_j(\tau_k)]} \Delta_i, \qquad (2.9) \\
0 = \frac{\partial}{\partial \alpha_j} l(\boldsymbol{\theta}) \quad &= \quad \frac{n_{\oplus j}}{\alpha_j} + \sum_{i=1}^{k} \sum_{l=1}^{n_{i\oplus}} \frac{\log G_j(t_{i;l})}{\alpha_j} \delta(c_{i;l} = j) - \sum_{i=1}^{k} \sum_{l=1}^{n_{i\oplus}} \frac{G_j(t_{i;l})}{1 - G_j(t_{i;l})} \frac{\log G_j(t_{i;l})}{\alpha_j} \delta(c_{i;l} \neq j) \\
&\quad - (n - n_{\oplus\oplus}) \frac{G_j(\tau_k)}{1 - G_j(\tau_k)} \frac{\log G_j(\tau_k)}{\alpha_j} \qquad (2.10)
\end{aligned}$$

for $1 \leq i \leq k$ and $1 \leq j \leq r$ where

$$U_{ij} = \Delta_i \sum_{i'=i+1}^{k} n_{i'j} + \sum_{l=1}^{n_{i\oplus}} (t_{i;l} - \tau_{i-1}) \delta(c_{i;l} = j) \qquad (2.11)$$

and $\delta(\cdot)$ is an indicator function that takes on the value of 1 if the argument is true and 0 otherwise. Note that U_{ij} in (2.11) is precisely the *Total Time on Test* statistic at the stress level x_i for the risk factor j. The MLEs $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\lambda}}, \widehat{\boldsymbol{\alpha}})$ are then obtained as simultaneous solutions to the above system of nonlinear equations. There is no closed form solution to the above equations and thus, some iterative search procedures such as the bisection method, Newton-Raphson method or Brent's method should be used to find the numerical solutions. As noted by Kalbfleisch (1985), the Newton-Raphson algorithm is often a convenient method to obtain the MLEs when there are two or more unknown parameters. The method works well if the likelihood is close to normal in shape. The asymptotic likelihood theory ensures the normal shape for large sample sizes, in which case this method is expected to be efficient; see Kalbfleisch (1980). Further analysis of the likelihood equations in (2.9) reveals that the MLE of λ_{ij} does not exist if $n_{ij} = 0$. That is, at least one failure caused by each risk factor must be observed at each stress level in order to estimate $\boldsymbol{\lambda}$ simultaneously. Consequently, the acceptable sample size needs to be much larger than the product of the number of stress levels implemented and the number of fatal risk factors under consideration in the planning stage of the experiment.

Remark 2.1. *In the model considered above, we have not assumed any relationships among the scale parameters* $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$ *of each risk factor. A popular log-linear relationship between the stress level and the scale parameter was not assumed either since it can be too restrictive when the physical stress-response relationship is not clear. The objective here is to estimate the parameters of each risk factor at each stress level (i.e., a full model) in order to investigate and formulate a plausible stress-response relationship which can be tested in the subsequent stage of analysis and incorporated into a reduced model. In certain situations, however, we may know that some particular relationships hold among the scale parameters; for instance,* $\lambda_{ij} = \rho_j \lambda_{(i-1)j}$ *with known* ρ_j*. In that case, the MLE of* λ_{ij} *exists whenever at least one failure occurs by the risk factor* j *(viz.,* $n_{\oplus j} > 0$*). One can also use the likelihood ratio test statistic to test the multiple hypotheses* $H_0: \ \lambda_{ij} = \rho_j \lambda_{(i-1)j}$ *for specified* ρ_j*'s.*

Remark 2.2. *The model proposed in Section 2 accommodates multiple stress levels and multiple competing risks. In fact, the model under consideration is general since it includes its marginal models as special cases. For instance, when* $k = 2$*,* $j = 2$*, and* $\alpha_1 = \alpha_2 = 1$*, the failure of a test unit will be caused by one of two competing risks from the exponential distributions in a simple step-stress ALT under Type-I censoring, which was considered by Han and Balakrishnan (2010). Consequently, the distributional results derived above simply reduce to those obtained by Han and Balakrishnan (2010) when* $k = 2$*,* $j = 2$*, and* $\alpha_1 = \alpha_2 = 1$*. On the other hand, when* $\boldsymbol{\lambda}_{\star j} \to \mathbf{0}_k$ *for* $j = 2, 3, \ldots, r$*, the failure of a test unit will be caused by a single risk factor with probability* 1*. Hence, the limiting case of the proposed model is the step-stress model under Type-I censoring without the competing risk structure. If we rather let* $\tau_1 \to \infty$*, then the model developed here converges to the ordinary single stress model (i.e., one stress level only) with multiple competing risks.*

3 Interval Estimations

In this section, we discuss the methods of constructing confidence intervals (CIs) for the unknown parameters $\boldsymbol{\theta}$. Since there is no closed form solution to the likelihood equations given in (2.9) and (2.10), it is not possible to derive the exact distributions of the MLEs. Hence, we construct the approximate CIs for the parameters based on the asymptotic distributions of the estimators, and also present the CIs using the parametric bootstrap approach for the purpose of comparison in simulation studies in Section 4.

3.1 *Approximate confidence intervals*

As shown in Section 2, $\widehat{\boldsymbol{\theta}}$ is non-linear functions of random quantities, which make it virtually impossible to find their exact marginal/joint distributions for exact inference. Hence, statistical inference for $\boldsymbol{\theta}$ is based on the asymptotic distributional result of the MLEs. As the sample size grows, the MLEs exhibit some special characteristics which are asymptotically optimal. First of all, under certain regularity conditions, the MLEs are asymptotically unbiased and efficient. That is, their biases tend to zero and their variances achieve the Cramer-Rao lower bound as the sample size goes to infinity. Furthermore, their distributions approach normal with the variance-covariance matrix given by the inverse of the Fisher information matrix $\boldsymbol{I}_n(\boldsymbol{\theta})$. Thus, inference about the unknown parameters can be based on the asymptotic normality of the MLEs that the vector $\widehat{\boldsymbol{\theta}}$ is approximately distributed as a multivariate normal with mean vector $\boldsymbol{\theta}$ and variance-covariance matrix $\boldsymbol{I}_n^{-1}(\boldsymbol{\theta})$. In this subsection, we present an approximate method to construct the CIs for $\boldsymbol{\theta}$ using these properties of the MLEs with large sample sizes. As noted by Balakrishnan and Han (2008), Han and Balakrishnan (2010), the approximate method provides not only the computational ease but also good probability coverage (close to the nominal level) when the sample size gets larger. This finding is further discussed in Section 4.

Let us first denote the (expected) Fisher information matrix of $\boldsymbol{\theta}$ by

$$\boldsymbol{I}_{E}(\boldsymbol{\theta}) = E\left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta_1 \partial\theta_2} \right]_{\theta_1,\theta_2\in\Omega} \tag{3.1}$$

where $\Omega = \{\lambda_{ij}, \alpha_j\}_{1\le i\le k,\ 1\le j\le r}$ is the complete set of the model parameters. The second partials of the log-likelihood in (2.8) are expressed as

$$\begin{aligned}
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}^2} &= \frac{n_{ij}}{\lambda_{ij}^2} + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1)\frac{1-\big[G_j(t_{i';l})\big]^{1/\alpha_j}}{\big[G_j(t_{i';l})\big]^{2/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]^2 \delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})\Big[\alpha_j\Big(1-\big[G_j(t_{i';l})\big]^{1/\alpha_j}\Big) - \Big(1-G_j(t_{i';l})\Big)\Big]}{\lambda_{i'j}\big[G_j(t_{i';l})\big]^{1/\alpha_j}\big[1-G_j(t_{i';l})\big]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]^2 \delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)\Big[\alpha_j\Big(1-\big[G_j(\tau_k)\big]^{1/\alpha_j}\Big) - \Big(1-G_j(\tau_k)\Big)\Big]}{\lambda_{kj}\big[G_j(\tau_k)\big]^{1/\alpha_j}\big[1-G_j(\tau_k)\big]^2}\Delta_i^2, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}\partial\lambda_{i''j}} &= \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1)\frac{1-\big[G_j(t_{i';l})\big]^{1/\alpha_j}}{\big[G_j(t_{i';l})\big]^{2/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\Big[\Delta_{i''}\delta(i'>i'') \\
&\quad + (t_{i'';l}-\tau_{i''-1})\delta(i'=i'')\Big]\delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})\Big[\alpha_j\Big(1-\big[G_j(t_{i';l})\big]^{1/\alpha_j}\Big) - \Big(1-G_j(t_{i';l})\Big)\Big]}{\lambda_{i'j}\big[G_j(t_{i';l})\big]^{1/\alpha_j}\big[1-G_j(t_{i';l})\big]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\Big[\Delta_{i''}\delta(i'>i'') + (t_{i'';l}-\tau_{i''-1})\delta(i'=i'')\Big]\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)\Big[\alpha_j\Big(1-\big[G_j(\tau_k)\big]^{1/\alpha_j}\Big) - \Big(1-G_j(\tau_k)\Big)\Big]}{\lambda_{kj}\big[G_j(\tau_k)\big]^{1/\alpha_j}\big[1-G_j(\tau_k)\big]^2}\Delta_i\Delta_{i''}, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}\partial\alpha_j} &= -\sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{1-\big[G_j(t_{i';l})\big]^{1/\alpha_j}}{\big[G_j(t_{i';l})\big]^{1/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})[1-G_j(t_{i';l}) + \log G_j(t_{i';l})]}{\alpha_j\lambda_{i'j}\big[1-G_j(t_{i';l})\big]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)[1-G_j(\tau_k)+\log G_j(\tau_k)]}{\alpha_j\lambda_{kj}\big[1-G_j(\tau_k)\big]^2}\Delta_i, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha_j^2} &= \frac{n_{\oplus j}}{\alpha_j^2} + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{G_j(t_{i';l})}{\big[1-G_j(t_{i';l})\big]^2}\left[\frac{\log G_j(t_{i';l})}{\alpha_j}\right]^2 \delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{G_j(\tau_k)}{\big[1-G_j(\tau_k)\big]^2}\left[\frac{\log G_j(\tau_k)}{\alpha_j}\right]^2,
\end{aligned}$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}\partial\lambda_{ij''}} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}\partial\lambda_{i''j''}} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\lambda_{ij}\partial\alpha_{j''}} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha_j\partial\alpha_{j''}} = 0$$

for $1 \leq i, i'' \leq k$ and $1 \leq j, j'' \leq r$ with $i \neq i''$ and $j \neq j''$. By substituting $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$, the elements of $\boldsymbol{I}_E(\boldsymbol{\theta})$ in (3.1) can be approximated by those of the observed Fisher information matrix given by

$$\boldsymbol{I}_O(\boldsymbol{\theta}) = \left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta_1\partial\theta_2} \right]_{\theta_1,\theta_2\in\Omega} \Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}. \tag{3.2}$$

Upon inverting this matrix in (3.2), we obtain the observed variance-covariance matrix of $\widehat{\boldsymbol{\theta}}$. Let $\theta \in \Omega$ and $\widehat{\theta}$ be the corresponding MLE of θ. Also, let V be the diagonal element of $\boldsymbol{I}_O^{-1}(\boldsymbol{\theta})$ corresponding to $\widehat{\theta}$. Since $\widehat{\theta}$ is asymptotically unbiased for θ, we can then use $(\widehat{\theta} - \theta)/\sqrt{V}$ as a pivotal quantity for θ to construct two-sided $100(1-\gamma)\%$ approximate CI for θ, which is given by

$$\left(\max\left\{0, \widehat{\theta} - z_{\gamma/2}\sqrt{V}\right\},\ \widehat{\theta} + z_{\gamma/2}\sqrt{V} \right) \tag{3.3}$$

where $z_{\gamma/2}$ is the upper $\gamma/2$-th quantile of a standard normal distribution.

3.2 *Bootstrap confidence intervals*

In this subsection, we construct the CIs for $\boldsymbol{\theta}$ using a parametric bootstrap method, *viz.*, the bias-corrected and accelerated (BCa) percentile bootstrap method; see Efron (1987), Hall (1988), and Efron and Tibshirani (1993) for details. Compared to the ordinary percentile bootstrap intervals or the Studentized-t bootstrap intervals, the BCa percentile bootstrap intervals are known to perform better. Kundu *et al.* (2004) also observed that the nonparametric bootstrap method does not work well for competing risks data. Before we obtain the BCa percentile bootstrap CIs for $\boldsymbol{\theta}$, the following algorithm is implemented to generate the bootstrap sample of size B based on the original Type-I censored sample of size $n_{\oplus\oplus}$:

Step 1 Given the stress change time points $\tau_1, \tau_2, \ldots, \tau_{k-1}$, the right censoring time point τ_k, the initial sample size n, and the original Type-I censored sample of size $n_{\oplus\oplus}$, calculate $\widehat{\boldsymbol{\theta}}$ by solving the system of the likelihood equations in (2.9) and (2.10).

Step 2 Generate a random sample of $\boldsymbol{U} = (U_1, U_2, \ldots, U_r)$ of size n, where U_j's are independently from the standard uniform distribution with the range $(0, 1)$. Set the counter $i = 1$ and $\eta = n$.

Step 3 Transform each $\boldsymbol{U} = (U_1, U_2, \ldots, U_r)$ in the sample into a vector $(T_{i1}, T_{i2}, \ldots, T_{ir})$ via

$$T_{ij} = -\frac{1}{\widehat{\lambda}_{ij}} \left[\log\left(1 - U_j^{1/\widehat{\alpha}_j}\right) + \sum_{l=1}^{i-1} \widehat{\lambda}_{lj} \Delta_l \right] + \tau_{i-1}$$

for $j = 1, 2, \ldots, r$ so that T_{ij} is a shifted generalized exponential variate with the scale parameter $\widehat{\lambda}_{ij}$ and the shape parameter $\widehat{\alpha}_j$. For each vector of $(T_{i1}, T_{i2}, \ldots, T_{ir})$, take the minimum of the elements as well as the corresponding index of the minimum (*e.g.*, record 3 if T_{i3} is the smallest). Let $\boldsymbol{T}_i$ be the vector of the minima collected and $\boldsymbol{C}_i$ be the vector of the corresponding indices, both of the dimension η.

Step 4 Sort the elements of $\boldsymbol{T}_i$ in an ascending order and permute the elements of $\boldsymbol{C}_i$ in a corresponding manner. Let $v_{1:\eta} < v_{2:\eta} < \cdots < v_{\eta:\eta}$ denote the ordered elements of $\boldsymbol{T}_i$ and let $w_1, w_2, \ldots, w_\eta$ denote the corresponding elements of $\boldsymbol{C}_i$.

Step 5 Find $n^*_{i\oplus}$ such that $v_{n^*_{i\oplus}:n} < \tau_i \leq v_{n^*_{i\oplus}+1:n}$. Then, for $1 \leq l \leq n^*_{i\oplus}$, set $t^*_{i;l}$ to be the value of $v_{l:\eta}$ and set $c^*_{i;l}$ to be the value of w_l. Also, set n^*_{ij} to be the number of j's in the first $n^*_{i\oplus}$ elements of the permuted $\boldsymbol{C}_i$ for $j = 1, 2, \ldots, r$ so that $\displaystyle\sum_{j=1}^{r} n^*_{ij} = n^*_{i\oplus}$.

Step 6 From the sample of $\boldsymbol{U}$ of size η, remove $\boldsymbol{U}$'s corresponding to each $t^*_{i;l}$ for $1 \leq l \leq n^*_{i\oplus}$ so that the reduced sample now has the new size of $\eta = n - \sum_{l=1}^{i} n^*_{l\oplus}$. Update the counter $i = i + 1$ and repeat Steps 3–6 until i hits $k + 1$.

Step 7 Define $n^*_{\oplus j} = \displaystyle\sum_{i=1}^{k} n^*_{ij}$ and $n^*_{\oplus\oplus} = \displaystyle\sum_{i=1}^{k} n^*_{i\oplus} = \sum_{j=1}^{r} n^*_{\oplus j}$. Based on $\tau_1, \tau_2, \ldots, \tau_k$, n, n^*_{ij}'s and the ordered observations $\boldsymbol{t}^* = (t^*_{1;1}, t^*_{1;2}, \ldots, t^*_{k;n^*_{k\oplus}})$ with the corresponding vector of the cause $\boldsymbol{c}^* = (c^*_{1;1}, c^*_{1;2}, \ldots, c^*_{k;n^*_{k\oplus}})$, calculate the new MLEs of $\boldsymbol{\theta}$, denoted by $\widehat{\boldsymbol{\theta}}^*$ from (2.9) and (2.10).

Step 8 Repeat Steps 2–7 B times. Then, for each $\theta \in \Omega$, arrange all the values of $\widehat{\theta}^*$ in an ascending order to obtain the bootstrap sample

$$\left\{\widehat{\theta}^{*[1]} < \widehat{\theta}^{*[2]} < \cdots < \widehat{\theta}^{*[B]}\right\}.$$

With the bootstrap samples generated as above, we now obtain the two-sided $100(1-\gamma)\%$ BCa percentile bootstrap CI for $\theta \in \Omega$ as

$$\left(\widehat{\theta}^{*[\beta_1 B]},\ \widehat{\theta}^{*[\beta_2 B]}\right)$$

where

$$\beta_1 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 - z_{\gamma/2}}{1-\widehat{a}\big[\widehat{z}_0 - z_{\gamma/2}\big]}\right) \qquad \text{and} \qquad \beta_2 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 + z_{\gamma/2}}{1-\widehat{a}\big[\widehat{z}_0 + z_{\gamma/2}\big]}\right).$$

Here, $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and the value of the bias-correction $\widehat{z}_0$ is given by

$$\widehat{z}_0 = \Phi^{-1}\left(\frac{\sum_{b=1}^{B}\delta\left(\widehat{\theta}^{*[b]} < \widehat{\theta}\right)}{B}\right)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal CDF and $\delta(\cdot)$ is an indicator function as defined before. A good estimate of the acceleration factor $\widehat{a}$ is suggested to be

$$\widehat{a} = \frac{\sum_{l=1}^{L}\left(\widehat{\theta}^{(l)} - \widehat{\theta}^{(\cdot)}\right)^3}{6\left\{\sum_{l=1}^{L}\left(\widehat{\theta}^{(l)} - \widehat{\theta}^{(\cdot)}\right)^2\right\}^{3/2}}$$

where

$$\widehat{\theta}^{(\cdot)} = \frac{1}{L}\sum_{l=1}^{L}\widehat{\theta}^{(l)}.$$

For $\theta = \lambda_{ij}$, $L = n_{ij}$ and $\widehat{\theta}^{(l)}$ is the MLE of λ_{ij} based on the original Type-I censored sample with the l-th observation deleted from the failures that occurred at the stress level x_i by the risk factor j (*i.e.*, the jackknife estimate) for $l = 1, 2, \ldots, n_{ij}$. Similarly, when $\theta = \alpha_j$, $L = n_{\oplus j}$ and $\widehat{\theta}^{(l)}$ is the MLE of α_j based on the original sample with the l-th observation deleted from the failures that occurred throughout the test by the risk factor j for $l = 1, 2, \ldots, n_{\oplus j}$.

4 Numerical Study

In order to evaluate the performance of the estimation methods discussed in the preceding section, an extensive Monte Carlo simulation study was conducted and the results are presented in this section. For the purpose of illustration, the case of the simple step-stress model (*viz.*, two stress levels, $k = 2$) with two competing risks (*viz.*, $r = 2$) is presented here since other various parameter settings we considered exhibited a similar pattern of the results. The values of the parameters were chosen to be $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, and $\alpha_2 = 2.0$. This particular scenario also describes a system with two components connected in series where the first component consists of three identical and independently operating sub-components connected in parallel while the second component consists of two identical and independently operating sub-components connected in parallel. The lifetime distribution of each sub-component is exponential with mean λ_{ij}^{-1}. Under the chosen parameter setting, the increased stress level causes 50% loss in the mean time to failure of any sub-component and the chance of a sub-component in the component 1 to fail is twice as high as the chance of a sub-component in the component 2 to fail before or after the change in the stress levels.

In order to explore the effects of several experimental parameters on the performance of estimation, the initial sample size n was chosen to be 25, 50, and 100, and several different choices were made for the stress change time point τ_1 while the censoring time point τ_2 was fixed at 1.0. Given the parameter values, a random sample was then generated by using Steps 2–7 of the algorithm for generating a bootstrap sample. Based on 1000 Monte Carlo simulations with $B = 1000$ bootstrap replications, the actual coverage probabilities of the 90%, 95%, and 99% intervals for each model parameter were determined empirically as well as the bias, relative absolute bias (RAB)[1], and MSE associated with the estimator. The results are presented in Tables 1-6 along with the estimated mean widths of the intervals from this simulation.

[1] $RAB = E\left[\left|\frac{\widehat{\theta} - \theta}{\theta}\right|\right]$ for $\theta \in \Omega$

Table 1: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 25$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
	0.3	1.057	0.911	2.726	84.0	87.6	88.5	93.2	94.6	97.8
λ_{11}	0.5	0.642	0.619	1.264	83.7	89.8	89.2	95.1	96.5	98.6
	0.7	0.330	0.353	0.923	90.4	89.1	94.3	97.0	98.0	98.5
	0.3	1.258	1.346	2.909	87.8	90.6	89.6	94.8	92.6	99.0
λ_{12}	0.5	0.723	0.887	1.287	85.9	87.9	89.5	92.9	93.3	96.9
	0.7	0.270	0.652	0.862	86.2	91.0	90.4	93.1	94.7	97.8
	0.3	0.140	0.242	1.518	88.7	87.6	93.6	94.8	99.0	98.8
λ_{21}	0.5	0.131	0.250	1.725	90.5	88.7	93.1	93.9	97.9	95.9
	0.7	0.490	0.388	1.928	86.9	88.8	93.0	92.5	97.5	96.6
	0.3	0.033	0.333	0.732	92.0	89.0	93.7	95.0	96.4	95.9
λ_{22}	0.5	0.112	0.404	1.074	88.1	92.0	92.9	96.3	95.2	99.5
	0.7	0.337	0.493	1.818	93.3	91.4	95.5	96.0	99.5	94.8
	0.3	2.658	1.862	9.258	88.2	90.5	89.3	93.5	93.2	97.8
α_1	0.5	1.957	1.058	5.877	89.6	87.9	93.0	94.0	94.5	96.6
	0.7	1.353	0.634	4.386	94.0	92.4	96.4	97.1	99.0	99.5
	0.3	2.819	1.973	9.549	90.5	90.7	94.6	94.6	95.0	98.7
α_2	0.5	2.080	1.670	8.312	90.3	87.9	93.0	93.1	95.0	97.2
	0.7	1.950	1.252	7.594	90.4	93.0	93.4	96.0	97.3	98.5

Table 2: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 25$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	8.330	9.160	9.353	11.021	11.205	13.419
	0.5	4.088	4.701	4.752	5.446	5.833	6.727
	0.7	2.801	3.663	3.321	4.240	4.249	4.814
λ_{12}	0.3	7.778	8.181	8.742	10.109	10.535	12.911
	0.5	3.718	5.389	4.193	6.305	5.009	7.000
	0.7	2.427	2.870	2.755	3.405	3.304	4.043
λ_{21}	0.3	3.668	4.922	4.364	5.709	5.698	6.607
	0.5	4.107	3.745	4.886	4.332	6.387	7.830
	0.7	5.788	6.874	6.846	7.710	8.703	8.734
λ_{22}	0.3	2.932	3.198	3.419	3.644	4.254	4.201
	0.5	3.268	3.603	3.804	4.231	4.686	4.973
	0.7	4.344	4.798	4.955	5.476	5.896	6.397
α_1	0.3	30.929	32.670	33.933	35.119	37.737	38.502
	0.5	17.777	19.742	22.310	22.887	27.016	29.174
	0.7	12.862	13.857	13.027	14.968	14.710	17.097
α_2	0.3	28.536	29.902	29.929	33.388	34.372	36.281
	0.5	24.689	25.115	27.713	30.698	30.444	32.629
	0.7	22.926	23.089	23.609	25.650	26.692	28.971

Table 3: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 50$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	1.032	0.872	2.406	84.6	94.1	88.6	97.0	95.6	98.5
	0.5	0.260	0.326	0.786	88.5	90.0	93.1	95.0	98.4	98.0
	0.7	0.170	0.251	0.419	83.4	89.6	91.0	93.8	98.0	97.8
λ_{12}	0.3	0.801	1.252	2.901	84.4	87.8	88.8	91.9	94.7	96.9
	0.5	0.322	0.632	0.797	87.0	90.0	92.2	95.2	96.5	99.5
	0.7	0.136	0.441	0.332	87.5	90.1	94.0	95.0	97.1	97.5
λ_{21}	0.3	0.086	0.157	0.665	88.0	90.5	94.2	95.5	99.0	98.0
	0.5	0.099	0.172	0.768	90.5	89.0	95.3	94.0	99.5	97.7
	0.7	0.222	0.228	1.373	93.1	88.8	95.0	92.8	98.1	96.8
λ_{22}	0.3	0.031	0.296	0.594	85.0	87.9	93.0	93.3	96.0	97.9
	0.5	0.111	0.297	0.551	89.3	87.6	93.6	93.5	97.4	96.9
	0.7	0.090	0.365	0.894	89.2	88.0	95.5	93.2	97.5	97.0
α_1	0.3	1.348	0.958	8.352	88.4	93.0	92.8	95.5	94.6	99.0
	0.5	0.949	0.502	5.077	92.0	88.6	95.0	93.6	97.3	97.7
	0.7	0.546	0.358	3.167	92.0	89.5	95.0	94.5	97.5	99.0
α_2	0.3	1.401	0.975	8.403	88.6	88.7	93.3	94.9	94.8	96.8
	0.5	0.936	0.666	7.856	92.0	91.0	94.0	95.0	97.5	99.0
	0.7	0.519	0.471	2.848	93.2	92.3	94.2	95.5	97.2	98.0

Table 4: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 50$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
	0.3	5.684	7.856	6.488	9.259	7.789	10.787
λ_{11}	0.5	2.740	3.619	3.251	4.209	4.155	4.784
	0.7	1.870	2.621	2.228	3.020	2.925	3.418
	0.3	4.278	5.841	4.810	7.039	5.771	9.025
λ_{12}	0.5	2.305	2.931	2.650	3.394	3.192	4.061
	0.7	1.667	1.973	1.951	2.251	2.416	2.570
	0.3	2.592	4.030	3.088	4.731	4.059	5.680
λ_{21}	0.5	2.793	4.555	3.328	5.452	4.373	6.188
	0.7	3.856	5.089	4.595	5.815	6.037	6.597
	0.3	2.279	2.713	2.692	3.143	3.435	3.558
λ_{22}	0.5	2.305	2.854	2.744	3.237	3.551	3.603
	0.7	2.904	3.402	3.418	3.562	4.275	4.010
	0.3	8.654	14.434	9.156	16.370	10.681	22.338
α_1	0.5	6.209	12.558	7.256	15.214	8.875	20.034
	0.7	4.172	5.991	4.964	7.347	6.460	9.127
	0.3	8.944	10.028	9.406	13.677	10.559	17.198
α_2	0.5	5.302	8.999	6.099	10.112	7.339	15.403
	0.7	3.744	7.591	4.381	9.181	5.441	11.231

Table 5: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 100$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	0.394	0.511	1.944	89.4	89.9	95.5	95.5	98.0	99.0
	0.5	0.135	0.240	0.374	89.5	93.0	93.9	96.2	98.4	97.9
	0.7	0.096	0.151	0.152	92.1	87.9	95.1	94.9	99.0	98.4
λ_{12}	0.3	0.517	0.913	2.281	88.9	89.5	94.5	95.3	97.8	98.5
	0.5	0.128	0.386	0.250	92.5	93.0	95.4	96.0	97.7	97.9
	0.7	0.092	0.306	0.152	89.0	90.2	93.9	94.6	96.9	99.1
λ_{21}	0.3	0.062	0.112	0.238	88.8	89.5	94.7	95.2	98.5	98.7
	0.5	0.080	0.115	0.326	93.1	88.9	96.1	95.0	100.0	98.0
	0.7	0.112	0.158	0.628	91.5	89.9	94.8	95.1	99.1	99.0
λ_{22}	0.3	0.030	0.206	0.283	88.7	89.0	95.3	95.1	97.9	98.6
	0.5	0.101	0.219	0.294	87.9	88.5	95.0	97.0	99.0	97.9
	0.7	0.115	0.244	0.393	89.2	90.1	96.0	96.0	98.8	99.5
α_1	0.3	1.054	0.577	6.840	90.5	89.7	95.5	93.9	97.9	99.1
	0.5	0.403	0.296	1.669	90.0	90.5	94.7	96.0	98.3	98.0
	0.7	0.245	0.213	0.711	93.2	89.8	98.0	94.8	99.0	96.9
α_2	0.3	1.328	0.876	5.248	89.0	89.6	94.7	94.7	98.7	98.9
	0.5	0.271	0.326	1.028	92.5	92.0	95.0	96.0	98.2	98.1
	0.7	0.222	0.257	0.503	90.0	89.9	94.8	93.8	98.5	99.5

Table 6: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 100$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
	0.3	3.677	4.598	4.285	5.213	5.265	5.890
λ_{11}	0.5	1.842	2.601	2.195	2.986	2.881	3.298
	0.7	1.299	2.116	1.548	2.545	2.034	2.887
	0.3	2.974	3.235	3.371	3.819	4.026	4.389
λ_{12}	0.5	1.568	1.909	1.847	2.182	2.309	2.472
	0.7	1.148	1.484	1.366	1.707	1.777	1.917
	0.3	1.834	3.217	2.186	4.035	2.873	4.976
λ_{21}	0.5	1.976	3.259	2.355	4.335	3.094	5.154
	0.7	2.657	4.338	3.166	4.975	4.161	5.691
	0.3	1.645	2.133	1.960	2.491	2.566	2.929
λ_{22}	0.5	1.612	2.117	1.921	2.539	2.523	2.984
	0.7	1.989	2.654	2.369	3.044	3.094	3.399
	0.3	6.894	12.849	7.948	13.994	9.690	14.835
α_1	0.5	3.435	4.791	4.093	5.680	5.374	6.596
	0.7	2.592	4.006	3.089	4.843	4.059	5.589
	0.3	6.654	8.458	7.626	10.447	9.297	12.952
α_2	0.5	2.672	4.251	3.160	5.402	4.073	6.523
	0.7	2.090	3.235	2.490	3.892	3.272	4.537

Although the crude approximate method based on the asymptotic normality of the MLEs is quick and easy, one major problem associated with it is that it does not necessarily take the parameter space into account when constructing CIs. There is no built-in procedure to prevent this and as a result, the lower bounds of the approximate CIs frequently hit below zero for small sample sizes or for high levels of confidence even though the parameters $\boldsymbol{\theta}$ can take only positive values in this setting. In order to turn such intervals into sensible ones, the negative lower bounds were all replaced by zero in the simulation according to (3.3).

From Tables 1, 3, and 5, we see that with fixed censoring time point τ_2, as the stress change time point τ_1 increases, the biases, RAB, MSE of the estimators for λ_{11}, λ_{12}, α_1, α_2 all decrease while those for λ_{21} and λ_{22} increase in most cases. The primary reason for this is that when Δ_1 gets larger or equivalently, when Δ_2 gets smaller with increasing τ_1, we expect a relatively large number of failures to occur before τ_1 (*i.e.*, at the first stress level), resulting in lower variability in the estimation of λ_{11} and λ_{12}. On the other hand, a relatively small number of failures will occur after τ_1 (*i.e.*, at the second stress level), resulting in higher variability in the estimation of λ_{21} and λ_{22}. The same intuition applies to explain the observation from Tables 2, 4, and 6 that with increasing τ_1, the widths of CIs for λ_{11}, λ_{12}, α_1, α_2 all decrease while those for λ_{21} and λ_{22} increase.

From Tables 1, 3, and 5, it is also observed that the biases, RAB, MSE of the estimators for α_j are mostly much higher than those for λ_{ij}, illustrating the difficulty of estimating the shape parameters with good accuracy and precision. This resulted in much wider CIs for α_j compared to those for λ_{ij} in Tables 2, 4, and 6. We also see that the MLEs overestimate the corresponding parameters on average since their biases are all positive with varying degrees. As the sample size n increases, however, the performance of the MLEs gets better as the biases, RAB, MSE associated with the estimators all decrease along with the actual coverage probabilities of the CIs getting closer to the nominal levels.

From Tables 2, 4, and 6, we observe that in comparison to the BCa bootstrap method, the approximate method consistently provides narrower intervals overall. This seems to explain somewhat better performance of the BCa bootstrap CIs relative to those obtained from the approximate method, especially for small sample sizes, as observed in Table 1. Nevertheless, we realize from Tables 3 and 5 that larger sample sizes eventually improve the actual coverage probabilities of both the approximate CIs and the BCa bootstrap CIs. As the sample size grows, a long computational time can be a problematic issue for constructing CIs by the bootstrap method. Hence, based on more exhaustive simulation study, it is recommended to use the BCa bootstrap approach for small sample sizes. When the initial sample size is considerably large, the approximate method is then computationally much easier to construct the intervals and they also perform quite well in terms of probability coverage for large sample sizes (*e.g.*, $n > 30$).

5 Illustrative Example

The dataset generated by Han and Balakrishnan (2010) is used here to illustrate the methods of inference described in the preceding sections. The dataset is a Type-I censored sample from a simple step-stress test with two known competing risks along with the stress change time point $\tau_1 = 3$ and the censoring time point $\tau_2 = 6$ for an equal step duration. It consists of total $n_{\oplus\oplus} = 23$ failure times from the initial sample size of $n = 25$ (*i.e.*, 8% right censoring). To be self-contained, the dataset is reproduced in Table 7 for easy reference.

From this dataset, we have $n_{11} = 7$, $n_{12} = 5$, $n_{21} = 5$, $n_{22} = 6$, and the observed MLEs of the GE parameters are estimated from (2.9) and (2.10) to be

$$\widehat{\lambda}_{11} = 0.085,\ \widehat{\lambda}_{12} = 0.167,\ \widehat{\lambda}_{21} = 0.229,\ \widehat{\lambda}_{22} = 0.373,\ \widehat{\alpha}_1 = 0.802,\ \widehat{\alpha}_2 = 1.548.$$

The observed standard errors of the estimators are obtained from (3.2) to be

$$\widehat{\sqrt{V}}_{\lambda_{11}} = 0.065,\ \widehat{\sqrt{V}}_{\lambda_{12}} = 0.125,\ \widehat{\sqrt{V}}_{\lambda_{21}} = 0.120,\ \widehat{\sqrt{V}}_{\lambda_{22}} = 0.154,\ \widehat{\sqrt{V}}_{\alpha_1} = 0.294,\ \widehat{\sqrt{V}}_{\alpha_2} = 0.799.$$

Table 7: Type-I censored sample from $n = 25$ units on a simple step-stress test with two competing risks, $\tau_1 = 3$ and $\tau_2 = 6$

Stress Level 1 (before $\tau_1 = 3$)		**Stress Level 2** (after $\tau_1 = 3$)	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.011	1	3.246	2
0.273	2	3.362	2
0.395	1	3.498	1
1.173	1	3.774	2
1.477	1	3.879	1
1.608	2	4.024	1
1.890	1	4.169	2
2.066	2	4.438	2
2.133	2	4.882	2
2.577	1	5.343	1
2.706	1	5.670	1
2.787	2		
$n_{1\oplus} = 12$		$n_{2\oplus} = 11$	

Table 8: Interval estimation based on the Type-I censored step-stress data in Table 7 with $B = 1000$

Parameter	CL	Approximate CI	BCa Bootstrap CI
λ_{11}	90%	(0.000, 0.191)	(0.018, 0.277)
	95%	(0.000, 0.212)	(0.014, 0.349)
	99%	(0.000, 0.252)	(0.009, 0.408)
λ_{12}	90%	(0.000, 0.372)	(0.027, 0.503)
	95%	(0.000, 0.412)	(0.019, 0.675)
	99%	(0.000, 0.489)	(0.010, 1.081)
λ_{21}	90%	(0.032, 0.426)	(0.080, 0.459)
	95%	(0.000, 0.464)	(0.072, 0.546)
	99%	(0.000, 0.538)	(0.048, 0.800)
λ_{22}	90%	(0.119, 0.626)	(0.167, 0.739)
	95%	(0.070, 0.675)	(0.138, 0.901)
	99%	(0.000, 0.770)	(0.081, 0.940)
α_1	90%	(0.319, 1.285)	(0.477, 2.031)
	95%	(0.226, 1.377)	(0.457, 3.049)
	99%	(0.045, 1.558)	(0.410, 3.970)
α_2	90%	(0.235, 2.862)	(0.785, 7.697)
	95%	(0.000, 3.114)	(0.732, 9.923)
	99%	(0.000, 3.606)	(0.646, 12.090)

Since the true parameter values are given as

$$\lambda_{11} = 0.112,\ \lambda_{12} = 0.082,\ \lambda_{21} = 0.223,\ \lambda_{22} = 0.246,\ \alpha_1 = \alpha_2 = 1,$$

the observed biases of the estimates are

$$\text{Bias}_{\lambda_{11}} = -0.027,\ \text{Bias}_{\lambda_{12}} = 0.085,\ \text{Bias}_{\lambda_{21}} = 0.006,\ \text{Bias}_{\lambda_{22}} = 0.126,\ \text{Bias}_{\alpha_1} = -0.198,\ \text{Bias}_{\alpha_2} = 0.548.$$

The CI for each parameter is also presented in Table 8 using the methods described in Section 3.

From Table 8, we observe that relative to the BCa bootstrap CIs, the approximate method consistently provides narrower CIs although every CI contains the respective true parameter value in this example. As expected, the CIs get wider as the nominal level of confidence increases. It is also observed that the lower bounds of the approximate CIs are zero, especially for small λ_{ij}'s and/or for high levels of confidence since they were hitting below zero in such cases. We also make a general observation from Table 8 that the CIs for the scale parameters are narrower than those for the shape parameters. Since the CIs for α_j's contain one, there is no sufficient statistical evidence to reject $H_0 :\ \alpha_j = 1$ with at most 10% level of significance. That is, the lifetime distribution of each risk factor is exponential, which is the correct decision in this situation.

6 Concluding Remarks

In this article, we have discussed the step-stress model under Type-I censoring when the lifetimes corresponding to different risk factors have independent GE distributions. Under the assumption of a cumulative damage model, point and interval estimations of the model parameters were discussed using the maximum likelihood approach. We have then conducted a simulation study to assess the performance of all these procedures and a numerical example has been presented to illustrate the methods of inference developed in this article. In the case of moderate to large sizes, the estimators give relatively accurate estimation of the parameters. Based on the results of the simulation study, our recommendation for constructing CIs for $\theta \in \Omega$ is to use the BCa percentile bootstrap method, especially in the case of small sample sizes. For larger sample sizes, however, the approximate method is more appropriate because of their computational ease as well as their improved probability coverage being close to the nominal levels. Based on our best knowledge, this study is the first to introduce the Type-I censoring to the GE distribution under the competing risk framework on the step-stress ALT. For future research, we will develop the Bayesian estimation method and a procedure for discriminating between the Weibull and GE distributions. We will also explore the extension of the methods developed for other distributions such as two-parameter Birnbaum-Saunders distribution discussed in Wang *et al.* (2006).

References

Abdel-Hamida, A.H., AL-Hussaini, E.K., 2009. Estimation in step-stress accelerated life tests for the exponentiated exponential distribution with type-I censoring. Comput. Statist. Data Anal. 53, 1328–1338.

Ateya, S.F., 2012. Maximum likelihood estimation under a finite mixture of generalized exponential distributions based on censored data. Statist. Papers (DOI: 10.1007/s00362-012-0480-z).

Bagdonavicius, V., 1978. Testing the hypothesis of additive accumulation of damages. Probab. Theory Application 23, 403–408.

Bagdonavicius, V., Nikulin, M., 2002. Accelerated Life Models: Modeling and Statistical Analysis. Chapman & Hall, Boca Raton, FL.

Balakrishnan, N., 2009. A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. Metrika 69, 351–396.

Balakrishnan, N., Han, D., 2008. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. J. Statist. Plann. Inference 138, 4172–4186.

Balakrishnan, N., Han, D., 2009. Optimal step-stress testing for progressively Type-I censored data from exponential distribution. J. Statist. Plann. Inference 139, 1782–1798.

Chen, D.G., Lio, Y.L., 2010. Parameter estimations for generalized exponential distribution under progressive Type-I interval censoring. Comput. Statist. Data Anal. 54, 1581–1591.

Cox, D.R., 1959. The analysis of exponentially distributed lifetimes with two types of failures. J. R. Stat. Soc. 21, 411–421.

Crowder, M.J., 2001. Classical Competing Risks. Chapman & Hall, Boca Raton, FL.

David, H.A., Moeschberger, M.L., 1978. The Theory of Competing Risks. Griffin, London.

Efron, B., 1987. Better bootstrap confidence intervals. J. Amer. Statist. Assoc. 82, 171–185.

Efron, B., Tibshirani, R., 1993. An Introduction to the Bootstrap. Chapman & Hall, New York.

Gouno, E., Balakrishnan, N., 2001. Step-stress accelerated life test. In: Balakrishnan, N., Rao, C.R. (Eds.), Handbook of Statistics 20: Advances in Reliability. North-Holland, Amsterdam, 623–639.

Gouno, E., Sen, A., Balakrishnan, N., 2004. Optimal step-stress test under progressive Type-I censoring. IEEE Trans. Reliab. 53, 383–393.

Gupta, R.D., Kundu, D., 2001. Generalized exponential distribution: different method of estimations. J. Statist. Comput. Simul. 69, 315–337.

Gupta, R.D., Kundu, D., 2003. Discriminating between Weibull and generalized exponential distributions. Comput. Statist. Data Anal. 43, 179–196.

Gupta, R.D., Kundu, D., 2004. Discriminating between gamma and generalized exponential distributions. J. Statist. Comput. Simul. 74, 107–121.

Gupta, R.D., Kundu, D., 2007. Generalized exponential distribution: existing results and some recent developments. J. Statist. Plann. Inference 137, 3537–3547.

Hall, P., 1988. Theoretical comparison of bootstrap confidence intervals. Ann. Statist. 16, 927–953.

Han, D., Balakrishnan, N., 2010. Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. Comput. Statist. Data Anal. 54, 2066–2081.

Han, D., Balakrishnan, N., Sen, A., Gouno, E., 2006. Corrections on "Optimal step-stress test under progressive Type-I censoring." IEEE Trans. Reliab. 55, 613–614.

Han, D., Ng, H.K.T., 2013. Asymptotic comparison between constant-stress testing and step-stress testing for Type-I censored data from exponential distribution. Comm. Statist. Theory Methods (revised).

Ismail, A.A., 2012. Inference in the generalized exponential distribution under partially accelerated tests with progressive Type-II censoring. Theor. Appl. Fract. Mech. 59, 49–56.

Kalbfleisch, J.G., 1980. The Statistical Analysis of Failure Time Data. Wiley, New York.

Kalbfleisch, J.G., 1985. Probability and Statistical Inference II. Springer, New York.

Klein, J.P., Basu, A.P., 1981. Weibull accelerated life tests when there are competing causes of failure. Comm. Statist. Theory Methods 10, 2073–2100.

Klein, J.P., Basu, A.P., 1982. Accelerated life tests under competing Weibull causes of failure. Comm. Statist. Theory Methods 11, 2271–2286.

Kundu, D., Kannan, N., Balakrishnan, N., 2003. Analysis of progressively censored competing risks data. In: Balakrishnan, N., Rao, C.R. (Eds.), Handbook of Statistics 23: Advances in Survival Analysis. Elsevier, New York, 331–348.

Meeker, W.Q., Escobar, L.A., 1998. Statistical Methods for Reliability Data. Wiley, New York.

Mudholkar, G.S., Srivastava, D.K., 1993. Exponentiated Weibull family for analyzing bathtub failure data. IEEE Trans. Reliab. 42, 299–302.

Nelson, W., 1980. Accelerated life testing: step-stress models and data analysis. IEEE Trans. Reliab. 29, 103–108.

Nelson, W., 1990. Accelerated Testing: Statistical Models, Test Plans, and Data Analyses. Wiley, New York.

Nelson, W., Meeker, W.Q., 1978. Theory for optimum accelerated censored life tests for Weibull and extreme value distributions. Technometrics 20, 171–177.

Raqab, M., Madi, M., 2005. Bayesian inference for the generalized exponential distribution. J. Stat. Comput. Simul. 75, 841–852.

Sedyakin, N.M., 1966. On one physical principle in reliability theory (in Russian). Techn. Cybernetics 3, 80–87.

Wang, Z., Desmond, A.F., Lu, X., 2006. Modified censored moment estimation for the two-parameter Birnbaum-Saunders distribution. Comput. Statist. Data Anal. 50, 1033–1051.