THE UNIVERSITY OF TEXAS AT SAN ANTONIO, COLLEGE OF BUSINESS

Working Paper SERIES

Date December 13, 2010

WP # 0017MSS-253-2010

Linear Models for Multivariate Repeated Measures Data

Anuradha Roy
Management Science and Statistics
University of Texas at San Antonio
San Antonio, Texas 78249
United States

ONE UTSA CIRCLE
SAN ANTONIO, TEXAS 78249-0631
210 458-4317 | BUSINESS.UTSA.EDU

Linear Models for Multivariate Repeated Measures Data

Anuradha Roy
Department of Management Science and Statistics
The University of Texas at San Antonio
San Antonio, Texas 78249, USA

Abstract

We study the general linear model (GLM) with doubly exchangeable distributed error for m observed random variables. The doubly exchangeable linear model (DEGLM) arises when the $m-$dimensional error vectors are "doubly exchangeable" (defined later), jointly normally distributed, which is much weaker assumption than the independent and identically distributed error vectors as in the case of GLM or classical GLM (CGLM). We estimate the parameters in the model and also find their distributions.

Key Words: Multivariate repeated measures; Linear model; Replicated observations.

JEL Classification: C10, C13

1 Introduction

A generalization of the general linear model (GLM) or the classical general linear model (CGLM) is considered by Arnold in 1979 when the $m \times 1$ error vectors are unobserved and exchangeable, jointly normally distributed; not independent and identically distributed (iid) as in the case of CGLM. He named his new model as exchangeable linear model (EGLM). EGLM is especially appropriate for doubly multivariate data or two-level multivariate data. The variance-covariance matrix $\mathbf{\Sigma}$ (partitioned) with exchangeable distributed error is of the form

$$\begin{aligned} \mathbf{\Sigma} &= \begin{bmatrix} \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 \\ \boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0 \end{bmatrix} \\ &= \mathbf{I}_u \otimes (\boldsymbol{U}_0 - \boldsymbol{U}_1) + \mathbf{J}_u \otimes \boldsymbol{U}_1, \end{aligned}$$

where $\mathbf{I}_u$ is the $u \times u$ identity matrix, $\mathbf{1}_u$ is a $u \times 1$ vector containing all elements as unity, $\mathbf{J}_u = \mathbf{1}_u \mathbf{1}_u', \boldsymbol{U}_0$ is a $m \times m$ positive definite symmetric matrix, and $\boldsymbol{U}_1$ is a symmetric $m \times m$ matrix. Leiva (2007) also used this variance-covariance matrix $\mathbf{\Sigma}$ for classification problems, and named this as equicorrelated partitioned matrix with equicorrelation parameters $\boldsymbol{U}_0, \boldsymbol{U}_1$. The $m \times m$ block diagonals $\boldsymbol{U}_0$ represent the variance-covariance matrix of the m response variables at any given site, whereas the $m \times m$ block off diagonals $\boldsymbol{U}_1$ represent the covariance matrix of the m response variables between any pair of sites. We assume $\boldsymbol{U}_0$ is constant for all sites. Also, $\boldsymbol{U}_1$ is constant between any pair of sites

In this article we extend Arnold's (1979) generalization of the EGLM when $m \times 1$ error vectors are unobserved and doubly exchangeable (defined in Section 2). Doubly exchangeable data is common in repeated measures designs in biomedical, medical, engineering, and in many other research areas. In repeated measures designs, in particular those employed in the clinical trial study of skin care products, the data are collected on a vector of measurements (m) at different body positions (u) and at different points (v) in time. For example, consider a clinical trial study where measurements are taken on the characteristics of wrinkling, pigmentation, inflammation, and hydration on hands, face, neck, and arms once in every month for four consecutive months. Occasionally, biomedical researchers measure levels of fat byproducts at different parts of the body (sites) in an eight-week clinical trial for their research. In other words, these data are multivariate in three levels. In these examples the variables at different sites and at different time points are not independent, but are stochastically dependent in nature. Different sites and different time points may be interchangeable or exchangeable (equicorrelated) among themselves; in other words it is reasonable to assume that the variables have doubly exchangeable structure. Doubly exchangeable linear model (DEGLM) is suitable for data that have doubly exchangeable structure.

In this article we develop DEGLM for three-level multivariate data by using doubly exchangeable structure or jointly equicorrelated covariance structure (Leiva, 2007; Roy and Leiva, 2007). Jointly equicorrelated covariance structure (defined in Section 2.1) assumes a block circulant covariance structure, consisting of three unstructured covariance matrices for three multivariate levels. This jointly equicorrelated covariance structure can capture double exchangeability in the data structure in a longitudinal study both in time and space. Another advantage of this covariance structure is that the measurements over time need not be of equally spaced.

Let $\boldsymbol{y}$ be the muv-variate vector of all measurements. We partition this vector $\boldsymbol{y}$ as follows:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_v \end{pmatrix}, \qquad \text{where} \quad \boldsymbol{y}_t = \begin{pmatrix} \boldsymbol{y}_{t1} \\ \vdots \\ \boldsymbol{y}_{tu} \end{pmatrix}, \qquad \text{with} \quad \boldsymbol{y}_{ts} = \begin{pmatrix} \boldsymbol{y}_{ts,1} \\ \vdots \\ \boldsymbol{y}_{ts,m} \end{pmatrix},$$

for $s = 1, \ldots, u$, $t = 1, \ldots, v$. The m-dimensional vector of measurements $\boldsymbol{y}_{ts}$ represents the replicate on the s^{th} location and at the t^{th} time point.

2 Basic results

2.1 Jointly equicorrelated vectors

Definition 1. *Let* $\boldsymbol{y}$ *be an* $muv-$*variate partitioned real-valued random vector* $\boldsymbol{y} = (\boldsymbol{y}'_1, \ldots, \boldsymbol{y}'_v)'$*, where* $\boldsymbol{y}_t = (\boldsymbol{y}'_{t1}, \ldots, \boldsymbol{y}'_{tu})'$ *for* $t = 1, \ldots, v$*, and* $\boldsymbol{y}'_{ts} = (y_{ts,1}, \ldots, y_{ts,m})'$ *for* $s = 1, \ldots, u$*. Let* $E[\boldsymbol{y}] = \boldsymbol{\mu_y} \in \Re^{muv}$*, and* $\boldsymbol{\Gamma_y}$ *be the* $(muv \times muv)-$*dimensional partitioned covariance matrix* $Cov[\boldsymbol{y}] = \left(\boldsymbol{\Gamma}_{\boldsymbol{y}_t, \boldsymbol{y}_{t^*}}\right) = (\boldsymbol{\Gamma}_{tt^*})$*, where* $\boldsymbol{\Gamma}_{tt^*} = Cov[\boldsymbol{y}_t, \boldsymbol{y}_{t^*}]$ *for* $t, t^* = 1, \ldots, v$*. The* $m-$*variate vectors* $\boldsymbol{y}_{11}, \ldots, \boldsymbol{y}_{1u}, \ldots, \boldsymbol{y}_{v1}, \ldots, \boldsymbol{y}_{vu}$ *are said to be jointly equicorrelated if* $\boldsymbol{\Gamma_y}$ *is given by*

$$\boldsymbol{\Gamma_y} = \mathbf{I}_{vu} \otimes (\mathbf{U}_0 - \mathbf{U}_1) + \mathbf{I}_v \otimes \mathbf{J}_u \otimes (\mathbf{U}_1 - \mathbf{W}) + \mathbf{J}_{vu} \otimes \mathbf{W}, \tag{1}$$

where $\mathbf{U}_0$ *is a positive definite symmetric* $m \times m$ *matrix, and* $\mathbf{U}_1$ *and* $\mathbf{W}$ *are symmetric* $m \times m$ *matrices. The variance covariance matrix* $\boldsymbol{\Gamma_y}$ *is then said to have a jointly equicorrelated covariance structure with equicorrelation parameters* $\mathbf{U}_0, \mathbf{U}_1$ *and* $\mathbf{W}$*. The matrices* $\mathbf{U}_0, \mathbf{U}_1$ *and* $\mathbf{W}$ *are all unstructured.*

Thus, the vectors $\boldsymbol{y}_{11}, \ldots, \boldsymbol{y}_{1u}, \ldots, \boldsymbol{y}_{v1}, \ldots, \boldsymbol{y}_{vu}$ are jointly equicorrelated if they have the following "jointly equicorrelated covariance" matrix

$$\boldsymbol{\Gamma_y} = \left[\begin{array}{cccc|cccc|c|cccc}
\mathbf{U}_0 & \mathbf{U}_1 & \cdots & \mathbf{U}_1 & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\mathbf{U}_1 & \mathbf{U}_0 & \cdots & \mathbf{U}_1 & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{U}_1 & \mathbf{U}_1 & \cdots & \mathbf{U}_0 & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\hline
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{U}_0 & \mathbf{U}_1 & \cdots & \mathbf{U}_1 & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{U}_1 & \mathbf{U}_0 & \cdots & \mathbf{U}_1 & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{U}_1 & \mathbf{U}_1 & \cdots & \mathbf{U}_0 & \cdots & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hline
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{U}_0 & \mathbf{U}_1 & \cdots & \mathbf{U}_1 \\
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{U}_1 & \mathbf{U}_0 & \cdots & \mathbf{U}_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \mathbf{W} & \mathbf{W} & \cdots & \mathbf{W} & \cdots & \mathbf{U}_1 & \mathbf{U}_1 & \cdots & \mathbf{U}_0
\end{array}\right], \tag{2}$$

that is,

$$\text{Cov}\left[\boldsymbol{y}_{ts}; \boldsymbol{y}_{t^*s^*}\right] = \begin{cases} \mathbf{U}_0 & \text{if} \quad t = t^* \quad \text{and} \quad s = s^*, \\ \mathbf{U}_1 & \text{if} \quad t = t^* \quad \text{and} \quad s \neq s^*, \\ \mathbf{W} & \text{if} \quad t \neq t^*, \end{cases}$$

The $m \times m$ block diagonals $\mathbf{U}_0$ in (2) represent the variance-covariance matrix of the m response variables at any given site and at any given time point, whereas the $m \times m$ block off diagonals $\mathbf{U}_1$ in (2) represent the covariance matrix of the m response variables between any two sites and at any given time point. We assume $\mathbf{U}_0$ is constant for all sites and time points, and $\mathbf{U}_1$ is the same for all site pairs and for all time points. The $m \times m$ block off diagonals $\mathbf{W}$ represent the covariance matrix of the m response variables between any two time points. It is assumed to be the same for any pair of time points, irrespective of the same site or between any two sites.

2.2 Matrix-variate normal distribution

The random matrix $\boldsymbol{X}(p \times n)$ is said to have a matrix-variate normal distribution with mean matrix $\boldsymbol{M}(p \times n)$ and covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{\Psi}$, where $\boldsymbol{\Sigma} > 0$, and $\boldsymbol{\Psi} > 0$ are $p \times p$ and $n \times n$ matrices respectively if and only if $\mathrm{Vec}(\boldsymbol{X}') \sim N_{pn}(\mathrm{Vec}(\boldsymbol{M}'), \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})$. We will use the notation $\boldsymbol{X} \sim N_{p,n}(\boldsymbol{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Psi})$ or $\boldsymbol{X} \sim N_{p,n}(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$. Note that, if $n = 1$ (thus, $\boldsymbol{\Psi}$ is a scalar), then $\boldsymbol{X}$ follows a $p-$ variate normal distribution with mean vector $\boldsymbol{M}$ and variance-covariance matrix $\boldsymbol{\Psi}\boldsymbol{\Sigma}$.

The matrix variate normal distribution arises when sampling from multivariate normal population. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be a random sample of size N from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Define the observation matrix as follows:

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pN} \end{bmatrix}$$

then $\boldsymbol{X}' \sim N_{N,p}(\mathbf{1}_N \boldsymbol{\mu}, \boldsymbol{I}_N \otimes \boldsymbol{\Sigma})$. We will use the following results of matrix-variate normal distribution (Pan and Fang, (2002); Gupta and Nagar, (2002)) in this article.

Result 1: $\boldsymbol{X} \sim N_{p,n}(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$, Then $\boldsymbol{X}' \sim N_{n,p}(\boldsymbol{M}', \boldsymbol{\Psi}, \boldsymbol{\Sigma})$.

Result 2: $\boldsymbol{X} \sim N_{p,n}(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$, and that $\boldsymbol{D}(m \times p)$ is of rank $m \leq p$, and $\boldsymbol{C}(n \times t)$ is of rank $t \leq n$, and $\boldsymbol{A}(m \times t)$, then $\boldsymbol{DXC} + \boldsymbol{A} \sim N_{m,t}(\boldsymbol{DMC} + \boldsymbol{A}, \boldsymbol{D\Sigma D}', \boldsymbol{C}'\boldsymbol{\Psi C})$.

Result 3: If $\boldsymbol{X}_1 \sim N_{p,n}(\boldsymbol{M}_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Psi}_1)$ and $\boldsymbol{X}_2 \sim N_{p,n}(\boldsymbol{M}_2, \boldsymbol{\Sigma}_2, \boldsymbol{\Psi}_2)$, then
$\boldsymbol{X}_1 + \boldsymbol{X}_2 \sim N_{p,n}\big(\boldsymbol{M}_1 + \boldsymbol{M}_2, (\boldsymbol{\Sigma}_1 \otimes \boldsymbol{\Psi}_1) + (\boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Psi}_2)\big)$.

2.3 Matrix results

The jointly equicorrelated variance-covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{y}}$ (Roy and Leiva, 2007) in (1) can be written as

$$\boldsymbol{\Gamma}_{\boldsymbol{y}} = \mathbf{I}_v \otimes (\mathbf{V}_0 - \mathbf{V}_1) + \mathbf{J}_v \otimes \mathbf{V}_1,$$

with

$$\mathbf{V}_0 = \mathbf{I}_u \otimes (\mathbf{U}_0 - \mathbf{U}_1) + \mathbf{J}_u \otimes \mathbf{U}_1,$$

and

$$\mathbf{V}_1 = \mathbf{J}_u \otimes \mathbf{W},$$

where $\mathbf{V}_0$ is a positive definite $(mu \times mu)-$dimensional symmetric matrix, $\mathbf{V}_1$ is a $(mu \times mu)-$dimensional symmetric matrix, and the $m \times m$ matrices $\mathbf{U}_0$, $\mathbf{U}_1$ and $\mathbf{W}$ are defined as in Section 2.1.

3 The Model

We study the doubly exchangeable general linear model (DEGLM) for three-level multivariate data by considering an $m \times u(v)$ dimensional random matrix $\boldsymbol{Y}$. What do I mean by this notation? The matrix has m rows, u columns and v depths. In other words, this notation means $m \times u$ dimensional matrices are stacked one after another v times. We now write the model as

$$\underset{m\times u(v)}{\boldsymbol{Y}} = \underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times u(v)}{\mathbf{1}'} + \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u(v)}{\boldsymbol{T}} + \underset{m\times u(v)}{\boldsymbol{e}},$$

or

$$\underset{u(v)\times m}{\boldsymbol{Y}'} = \underset{u(v)\times 1}{\mathbf{1}}\underset{1\times m}{\boldsymbol{\alpha}'} + \underset{u(v)\times r-1}{\boldsymbol{T}'}\underset{r-1\times m}{\boldsymbol{\gamma}} + \underset{u(v)\times m}{\boldsymbol{e}'}, \tag{3}$$

where $\boldsymbol{Y}$ is a $m \times u(v)$ dimensional random matrix. $\boldsymbol{\alpha}$ is an $m-$ dimensional vector. $\boldsymbol{\gamma}$ is a $(r-1) \times m$ matrix. $\boldsymbol{T}$ is an $(r-1) \times u(v)-$dimensional matrix such that the design matrix $\boldsymbol{X} = [\mathbf{1}, \boldsymbol{T}']$ has rank r. We assume $uv > r$. The error matrix $\boldsymbol{e}$ is such that the $m \times 1-$dimensional components of $\mathrm{Vec}(\boldsymbol{e})$, $\boldsymbol{e}_{11}, \ldots, \boldsymbol{e}_{1u}, \ldots, \boldsymbol{e}_{v1}, \ldots, \boldsymbol{e}_{vu}$ are doubly exchangeable, i.e., $\mathrm{E}(\boldsymbol{e}_{ts}) = \mathbf{0}$, for $s = 1, \ldots, u$, $t = 1, \ldots, v$, and

$$\mathrm{Cov}\left[\boldsymbol{e}_{ts}; \boldsymbol{e}_{t^*s^*}\right] = \begin{cases} \mathbf{U}_0 & \text{if} \quad t = t^* \quad \text{and} \quad s = s^*, \\ \mathbf{U}_1 & \text{if} \quad t = t^* \quad \text{and} \quad s \neq s^*, \\ \mathbf{W} & \text{if} \quad t \neq t^*, \end{cases}$$

Arnold (1979) showed that the usual methods for making inferences about $\boldsymbol{\alpha}$ in the CGLM are not valid for the EGLM. He also mentioned that it was difficult to extract much information about $\boldsymbol{U}_0$ from the data. Thus, there is no sensible way to test hypotheses about $\boldsymbol{\alpha}$ in the EGLM. Our model (3) is an improvement over Arnold's model as with $\boldsymbol{W} = 0$ and $\min(u, v) \geq m + r$ one can test $\boldsymbol{\alpha}$ in the EGLM.
To compute the model parameters and their distributions we first need to prove some lemmas.

Lemma 1. *Let* $\boldsymbol{\Gamma} = \underset{v\times v}{\boldsymbol{C}'} \otimes \boldsymbol{I}_{mu}$ *and* $\boldsymbol{\Gamma}^{\bullet} = \boldsymbol{I}_v \otimes (\underset{u\times u}{\boldsymbol{C}^{*\prime}} \otimes \boldsymbol{I}_m)$ *where* $\boldsymbol{C}$ *and* $\boldsymbol{C}^*$ *are orthogonal matrices whose first columns are proportional to* $\mathbf{1}$*'s. Let* $\boldsymbol{\Gamma}_{\boldsymbol{y}}$ *be a jointly equicorrelated covariance matrix as in equation* (2) *of Def. 1, then* $\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(\mathbf{\Gamma}_{\boldsymbol{y}})\mathbf{\Gamma}'\mathbf{\Gamma}^{\bullet\prime}$ *is a diagonal matrix as follows:*

$$\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(\mathbf{\Gamma}_{\boldsymbol{y}})[\mathbf{\Gamma}'\mathbf{\Gamma}^{\bullet\prime} = \begin{bmatrix} \mathbf{\Delta}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\mathbf{\Delta}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Delta}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\mathbf{\Delta}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Delta}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I}_{u-1}\otimes\mathbf{\Delta}_1 \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{\Delta}_1 &= \mathbf{U}_0 - \mathbf{U}_1, \\ \mathbf{\Delta}_2 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 - u\mathbf{W} = (\mathbf{U}_0 - \mathbf{U}_1) + u\,(\mathbf{U}_1 - \mathbf{W})\,, \quad \textit{and} \\ \mathbf{\Delta}_3 &= \mathbf{U}_0 + (u-1)\,\mathbf{U}_1 + u\,(v-1)\,\mathbf{W} = (\mathbf{U}_0 - \mathbf{U}_1) + u\,(\mathbf{U}_1 - \mathbf{W}) + uv\mathbf{W}. \end{aligned}$$

Proof: It can be easily shown that $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^{\bullet}$ are orthogonal. We see that

$$\begin{aligned} \mathbf{\Gamma}(\mathbf{\Gamma}_{\boldsymbol{y}})\mathbf{\Gamma}' &= (\underset{v\times v}{\boldsymbol{C}'}\otimes\boldsymbol{I}_{mu})(\mathbf{I}_v\otimes(\mathbf{V}_0-\mathbf{V}_1)+\mathbf{J}_v\otimes\mathbf{V}_1)(\underset{v\times v}{\boldsymbol{C}}\otimes\boldsymbol{I}_{mu}), \\ &= \mathbf{I}_v\otimes(\mathbf{V}_0-\mathbf{V}_1) + \begin{bmatrix} v & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \otimes \mathbf{V}_1, \\ &= \begin{bmatrix} \boldsymbol{V}_o + (v-1)\boldsymbol{V}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{v-1}\otimes(\boldsymbol{V}_o - \boldsymbol{V}_1) \end{bmatrix}. \end{aligned} \tag{4}$$

The determinant of $\mathbf{\Gamma}_{\boldsymbol{y}}$ is given by

$$|\mathbf{\Gamma}(\mathbf{\Gamma}_{\boldsymbol{y}})\mathbf{\Gamma}'| = |\mathbf{\Gamma}_{\boldsymbol{y}}| = |\boldsymbol{V}_o + (v-1)\boldsymbol{V}_1||\boldsymbol{V}_o - \boldsymbol{V}_1|^{v-1}.$$

Therefore, the matrix $\mathbf{\Gamma}_{\boldsymbol{y}}$ is non-singular, if both $\boldsymbol{V}_o + (v-1)\boldsymbol{V}_1$ and $\boldsymbol{V}_o - \boldsymbol{V}_1$ are non-singular matrices. Now, from (4), we have

$$\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(\mathbf{\Gamma}_{\boldsymbol{y}})\mathbf{\Gamma}'\mathbf{\Gamma}^{\bullet\prime} = \begin{bmatrix} (\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes\boldsymbol{I}_m)(\boldsymbol{V}_o+(v-1)\boldsymbol{V}_1)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes\boldsymbol{I}_m) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{v-1}\otimes(\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes\boldsymbol{I}_m)(\boldsymbol{V}_o-\boldsymbol{V}_1)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes\boldsymbol{I}_m) \end{bmatrix}. \tag{5}$$

Now,

$$\begin{aligned} (\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes\boldsymbol{I}_m)(\mathbf{V}_0+(v-1)\boldsymbol{V}_1)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes\boldsymbol{I}_m) &= (\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes\boldsymbol{I}_m)\big(\mathbf{I}_u\otimes(\mathbf{U}_0-\mathbf{U}_1)+\mathbf{J}_u\otimes[(\mathbf{U}_1-\mathbf{W})+v\mathbf{W}]\big)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes\boldsymbol{I}_m) \\ &= \begin{bmatrix} (\boldsymbol{U}_o-\boldsymbol{U}_1)+u(\boldsymbol{U}_1-\boldsymbol{W})+uv\boldsymbol{W} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes(\boldsymbol{U}_o-\boldsymbol{U}_1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{\Delta}_3 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\mathbf{\Delta}_1 \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned}(\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes \boldsymbol{I}_m)(\mathbf{V}_0-\boldsymbol{V}_1)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes \boldsymbol{I}_m) &= (\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes \boldsymbol{I}_m)\big(\mathbf{I}_u\otimes(\mathbf{U}_0-\mathbf{U}_1)+\mathbf{J}_u\otimes(\mathbf{U}_1-\mathbf{W})\big)(\underset{u\times u}{\boldsymbol{C}^{*}}\otimes \boldsymbol{I}_m)\\ &= \begin{bmatrix} (\boldsymbol{U}_o-\boldsymbol{U}_1)+u(\boldsymbol{U}_1-\boldsymbol{W}) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes(\boldsymbol{U}_o-\boldsymbol{U}_1)\end{bmatrix}\\ &= \begin{bmatrix} \boldsymbol{\Delta}_2 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes \boldsymbol{\Delta}_1\end{bmatrix}.\end{aligned}$$

Therefore, from (5) we have

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\boldsymbol{\Gamma}_{\boldsymbol{y}})\boldsymbol{\Gamma}'\boldsymbol{\Gamma}^{\bullet\prime} = \begin{bmatrix} \boldsymbol{\Delta}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 \end{bmatrix}.$$

It follows that if $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$ and $\boldsymbol{\Delta}_3$ are non-singular then $\boldsymbol{\Gamma}_{\boldsymbol{y}}$ is non-singular.

Corollary 1. *If* $\mathbf{W}=\mathbf{0}$*, then*

$$\begin{aligned}\boldsymbol{\Delta}_1 &= \mathbf{U}_0-\mathbf{U}_1,\\ \boldsymbol{\Delta}_2 &= \mathbf{U}_0+(u-1)\,\mathbf{U}_1=(\mathbf{U}_0-\mathbf{U}_1)+u\mathbf{U}_1\\ \textit{and}\quad \boldsymbol{\Delta}_3 &= \mathbf{U}_0+(u-1)\,\mathbf{U}_1=(\mathbf{U}_0-\mathbf{U}_1)+u\mathbf{U}_1.\end{aligned}$$

Thus, we see that $\boldsymbol{\Delta}_2=\boldsymbol{\Delta}_3$.

Lemma 2. *Let* $\boldsymbol{Y}$ *is a* $m\times u(v)$ *dimensional random matrix. Then* $\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec\underset{m\times u(v)}{\boldsymbol{Y}})$ *can be expressed as*

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec\underset{m\times u(v)}{\boldsymbol{Y}}) = \begin{bmatrix} Vec(\underset{m\times 1}{\boldsymbol{Z}_{11}}:\underset{m\times(u-1)}{\boldsymbol{Z}_{12}})\\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{21}}:\underset{m\times(u-1)}{\boldsymbol{Z}_{22}})\\ \vdots\\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{v1}}:\underset{m\times(u-1)}{\boldsymbol{Z}_{v2}})\end{bmatrix}.$$

Proof:

$$\begin{aligned}
\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\mathrm{Vec}\underset{m\times u(v)}{\boldsymbol{Y}}) &= \boldsymbol{\Gamma}^{\bullet}(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_u\otimes \boldsymbol{I}_m)Vec(\underset{m\times u(v)}{\boldsymbol{Y}}) \\
&= \begin{bmatrix} \mathrm{Vec}(\underset{m\times u}{\boldsymbol{Y}_1}\underset{u\times u}{\boldsymbol{C}^*}) \\ \mathrm{Vec}(\underset{m\times u}{\boldsymbol{Y}_2}\underset{u\times u}{\boldsymbol{C}^*}) \\ \vdots \\ \mathrm{Vec}(\underset{m\times u}{\boldsymbol{Y}_v}\underset{u\times u}{\boldsymbol{C}^*}) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Vec}(\frac{1}{\sqrt{u}}\underset{m\times u}{\boldsymbol{Y}_{11}}\mathbf{1}_u : \underset{m\times(u-1)}{\boldsymbol{Z}^*_{12}}) \\ \mathrm{Vec}(\frac{1}{\sqrt{u}}\underset{m\times u}{\boldsymbol{Y}_{21}}\mathbf{1}_u : \underset{m\times(u-1)}{\boldsymbol{Z}^*_{22}}) \\ \vdots \\ \mathrm{Vec}(\frac{1}{\sqrt{u}}\underset{m\times u}{\boldsymbol{Y}_{v1}}\mathbf{1}_u : \underset{m\times(u-1)}{\boldsymbol{Z}^*_{v2}}) \end{bmatrix} = \begin{bmatrix} \mathrm{Vec}(\underset{m\times 1}{\boldsymbol{Z}_{11}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{12}}) \\ \mathrm{Vec}(\underset{m\times 1}{\boldsymbol{Z}_{21}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{22}}) \\ \vdots \\ \mathrm{Vec}(\underset{m\times 1}{\boldsymbol{Z}_{v1})} : \underset{m\times(u-1)}{\boldsymbol{Z}_{v2}}) \end{bmatrix}
\end{aligned}$$

Lemma 3. $\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times uv}{\boldsymbol{T}}))$ *can be expressed as*

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times uv}{\boldsymbol{T}})) = Vec\Big(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \cdots : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{v2}}\Big).$$

Proof:

$$\begin{aligned}
\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times uv}{\boldsymbol{T}})) &= \boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(\mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times uv}{\boldsymbol{T}})) \\
&= \boldsymbol{\Gamma}^{\bullet}(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_{mu})(\mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}(\underset{r-1\times uv}{\boldsymbol{T}}))) \\
&= (\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes \boldsymbol{I}_m)(\mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_1} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_2} : \ldots : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_v}))) \\
&= \begin{bmatrix} \mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_1}\underset{u\times u}{\boldsymbol{C}^*}) \\ \mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_2}\underset{u\times u}{\boldsymbol{C}^*}) \\ \vdots \\ \mathrm{Vec}(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u}{\boldsymbol{T}_v}\underset{u\times u}{\boldsymbol{C}^*}) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Vec}\Big(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{12}}\Big) \\ \mathrm{Vec}\Big(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{21}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{22}}\Big) \\ \vdots \\ \mathrm{Vec}\Big(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{v2}}\Big) \end{bmatrix} \\
&= \mathrm{Vec}\Big(\underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \cdots : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{m\times r-1}{\boldsymbol{\gamma}'}\underset{r-1\times u-1}{\boldsymbol{U}_{v2}}\Big).
\end{aligned}$$

Lemma 4. *$\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(Vec(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'}))$ can be expressed as*

$$\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(Vec(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'})) = Vec\left(\ \sqrt{uv}\underset{m\times 1}{\boldsymbol{\alpha}}:\ \ \mathbf{0}:\ \ \cdots:\ \ \mathbf{0}\ \right).$$

Proof:

$$\begin{aligned}\mathbf{\Gamma}^{\bullet}\mathbf{\Gamma}(\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'})) &= \mathbf{\Gamma}^{\bullet}\big(\mathbf{\Gamma}(\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'}))\big)\\ &= \mathbf{\Gamma}^{\bullet}\big[(\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'})(\boldsymbol{C}\otimes\boldsymbol{I}_u))\big]\end{aligned}$$

We will first calculate $\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'}(\boldsymbol{C}\otimes\boldsymbol{I}_u))$. Now,

$$\begin{aligned}&(\boldsymbol{C}'\otimes\boldsymbol{I}_u)\underset{uv\times 1}{\mathbf{1}}\underset{1\times m}{\boldsymbol{\alpha}'}\\
&= \left(\begin{bmatrix}\frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}} & \cdots & \frac{1}{\sqrt{v}}\\ \frac{1}{\sqrt{1\times 2}} & \frac{-1}{\sqrt{1\times 2}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\sqrt{(v-1)\times v}} & \frac{1}{\sqrt{(v-1)\times v}} & \cdots & \frac{-(v-1)}{\sqrt{(v-1)\times v}}\end{bmatrix}\otimes\boldsymbol{I}_u\right)\begin{bmatrix}\underset{u\times 1}{\mathbf{1}}\\ \underset{u\times 1}{\mathbf{1}}\\ \vdots\\ \underset{u\times 1}{\mathbf{1}}\end{bmatrix}\underset{1\times m}{\boldsymbol{\alpha}'}\\
&= \begin{bmatrix}\sqrt{v}\underset{u\times 1}{\mathbf{1}}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}\underset{1\times m}{\boldsymbol{\alpha}'}\\
&= \begin{bmatrix}\sqrt{v}\underset{u\times 1}{\mathbf{1}}\underset{1\times m}{\boldsymbol{\alpha}'}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}\end{aligned}$$

Therefore,

$$\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'}(\boldsymbol{C}\otimes\boldsymbol{I}_u)) = \mathrm{Vec}\left(\ \sqrt{v}\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times u}{\mathbf{1}'}:\ \ \mathbf{0}:\ \ \cdots:\ \ \mathbf{0}\ \right).$$

Thus,

$$\begin{aligned}&\mathbf{\Gamma}^{\bullet}\big[(\mathrm{Vec}(\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times uv}{\mathbf{1}'})(\boldsymbol{C}\otimes\boldsymbol{I}_u))\big]\\
&(\boldsymbol{I}_v\otimes\underset{u\times u}{\boldsymbol{C}^{*\prime}}\otimes\boldsymbol{I}_m)\mathrm{Vec}\left(\ \sqrt{v}\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times u}{\mathbf{1}'}:\ \ \mathbf{0}:\ \ \cdots:\ \ \mathbf{0}\ \right)\\
&\mathrm{Vec}\left(\ \sqrt{v}\underset{m\times 1}{\boldsymbol{\alpha}}\underset{1\times u}{\mathbf{1}'}\underset{u\times u}{\boldsymbol{C}^{*}}:\ \ \mathbf{0}:\ \ \cdots:\ \ \mathbf{0}\ \right).\end{aligned}$$

Now, we see that

$$
\begin{aligned}
& \underset{u\times u}{\boldsymbol{C}^{*\prime}} \left[\ \sqrt{v}\ \underset{u\times 1}{\mathbf{1}}\ \right] \\
= & \begin{bmatrix} \sqrt{uv} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
& \mathrm{Vec}\left(\sqrt{v}\ \underset{m\times 1}{\boldsymbol{\alpha}}\ \underset{1\times u}{\mathbf{1}'}\ \underset{u\times u}{\boldsymbol{C}^{*}}\right) \\
= & \begin{bmatrix} \sqrt{uv}\ \underset{m\times 1}{\boldsymbol{\alpha}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},
\end{aligned}
$$

and the lemma is proved.

Lemma 5.

$$
(\boldsymbol{I}_v \otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\boldsymbol{C}' \otimes \boldsymbol{I}_u))\underset{uv\times 1}{\mathbf{1}} = \begin{bmatrix} \sqrt{uv} \\ \mathbf{0}_{u-1} \\ \mathbf{0}_u \\ \vdots \\ \mathbf{0}_u \end{bmatrix}
$$

Proof:

$$
\begin{aligned}
& (\boldsymbol{I}_v \otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\boldsymbol{C}' \otimes \boldsymbol{I}_u))\underset{uv\times 1}{\mathbf{1}} \\
= & (\boldsymbol{I}_v \otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}}) \left(\begin{bmatrix} \frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}} & \cdots & \frac{1}{\sqrt{v}} \\ \frac{1}{\sqrt{1\times 2}} & \frac{-1}{\sqrt{1\times 2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{(v-1)\times v}} & \frac{1}{\sqrt{(v-1)\times v}} & \cdots & \frac{-(v-1)}{\sqrt{(v-1)\times v}} \end{bmatrix} \otimes \boldsymbol{I}_u \right) \begin{bmatrix} \underset{u\times 1}{\mathbf{1}} \\ \underset{u\times 1}{\mathbf{1}} \\ \vdots \\ \underset{u\times 1}{\mathbf{1}} \end{bmatrix} \\
= & (\boldsymbol{I}_v \otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}}) \begin{bmatrix} \frac{1}{\sqrt{v}}\underset{u\times 1}{\mathbf{1}} + \frac{1}{\sqrt{v}}\underset{u\times 1}{\mathbf{1}} + \cdots + \frac{1}{\sqrt{v}}\underset{u\times 1}{\mathbf{1}} \\ \frac{1}{\sqrt{1\times 2}}\underset{u\times 1}{\mathbf{1}} + \frac{-1}{\sqrt{1\times 2}}\underset{u\times 1}{\mathbf{1}} + \cdots + 0 \\ \cdots \\ \frac{1}{\sqrt{(v-1)\times v}}\underset{u\times 1}{\mathbf{1}} + \frac{1}{\sqrt{(v-1)\times v}}\underset{u\times 1}{\mathbf{1}} + \cdots - \frac{(v-1)}{\sqrt{(v-1)\times v}}\underset{u\times 1}{\mathbf{1}} \end{bmatrix} \\
= & \begin{bmatrix} \sqrt{uv} \\ \mathbf{0}_{u-1} \\ \mathbf{0}_u \\ \vdots \\ \mathbf{0}_u \end{bmatrix}
\end{aligned}
$$

Theorem 1. *Let* $\underset{m\times u(v)}{\boldsymbol{Y}}$ *represent the three-level multivariate data. If*

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}(Vec \underset{m\times(u)v}{\boldsymbol{Y}}) = \begin{bmatrix} Vec(\underset{m\times 1}{\boldsymbol{Z}_{11}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{12}}) \\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{21}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{22}}) \\ \vdots \\ Vec(\underset{m\times 1}{\boldsymbol{Z}_{v1}} : \underset{m\times(u-1)}{\boldsymbol{Z}_{v2}}) \end{bmatrix}$$

then all the components $\boldsymbol{Z}_{11}$*,* $\boldsymbol{Z}_{12}$*,* $\boldsymbol{Z}_{21}, \ldots, \boldsymbol{Z}_{v2}$ *are independently normally distributed such that*

$$\boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{uv}\boldsymbol{\alpha} + \boldsymbol{\gamma}'\boldsymbol{U}_{11}, \boldsymbol{\Delta}_3, 1), \tag{6a}$$

$$\boldsymbol{Z}_{i1} \sim N_{m,1}(\boldsymbol{\gamma}'\boldsymbol{U}_{i1}, \boldsymbol{\Delta}_2, 1) \;\; i = 2, 3, \ldots, v, \tag{6b}$$

$$\text{and} \qquad \boldsymbol{Z}_{i2} \sim N_{m,u-1}(\boldsymbol{\gamma}'\boldsymbol{U}_{i2}, \boldsymbol{\Delta}_1, \boldsymbol{I}_{u-1}) \;\; i = 1, 2, \ldots, v. \tag{6c}$$

Proof: Follows from Lemmas 1 and 2.

Comments: Thus, we see that uv CGLMs are nested in one DEGLM. The model involving $\boldsymbol{Z}_{11}$ has covariance matrix $\boldsymbol{\Delta}_3$, but has only one sample, thus estimation of $\boldsymbol{\Delta}_3$ is not possible. Testing of $\boldsymbol{\alpha}$ is therefore not possible. This case is similar to Arnold's (1979) EGLM, where the intercept $\boldsymbol{\alpha}$ could not be tested. Each of the models involving $\boldsymbol{Z}_{i1}$ $i = 2, 3, \ldots, v$ has no intercept and has only one sample too, however, they have common variance-covariance matrix $\boldsymbol{\Delta}_2$. Thus, $\boldsymbol{\gamma}$ and $\boldsymbol{\Delta}_2$ can be estimated and $\boldsymbol{\gamma}$ can be tested too.

Similarly, each of the models involving $\boldsymbol{Z}_{i2}, i = 1, 2, \ldots, v$, are all independent and have a common covariance matrix $\boldsymbol{\Delta}_1$ and have no intercept term. Thus, one can calculate estimates of $\boldsymbol{\gamma}$ and $\boldsymbol{\Delta}_1$ from each of the models, and test any hypothesis about $\boldsymbol{\gamma}$.

Corollary 2. *If* $\boldsymbol{W} = \boldsymbol{0}$*, then*

$$\boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{uv}\boldsymbol{\alpha} + \boldsymbol{\gamma}'\boldsymbol{U}_{11}, \boldsymbol{\Delta}_2, 1),$$

$$\boldsymbol{Z}_{i1} \sim N_{m,1}(\boldsymbol{\gamma}'\boldsymbol{U}_{i1}, \boldsymbol{\Delta}_2, 1) \;\; i = 2, 3, \ldots, v,$$

$$\text{and} \qquad \boldsymbol{Z}_{i2} \sim N_{m,u-1}(\boldsymbol{\gamma}'\boldsymbol{U}_{i2}, \boldsymbol{\Delta}_1, \boldsymbol{I}_{u-1}) \;\; i = 1, 2, \ldots, v.$$

Comments: In this case there are v independent samples to estimate $\boldsymbol{\Delta}_2$. Thus, testing any hypothesis about $\boldsymbol{\alpha}$ is possible in the Arnold's EGLM when one use DEGLM with $\boldsymbol{W} = \boldsymbol{0}$ and $\min(u, v) \geq m + r$.

Corollary 3. *If* $v = 1$*, then the DEGLM reduces to the EGLM. For* $v = 1$*, we have*

$$\boldsymbol{Z}_{11} \sim N_{m,1}(\sqrt{u}\boldsymbol{\alpha} + \boldsymbol{\gamma}'\boldsymbol{U}_{11}, \boldsymbol{U}_0 - (u-1)\boldsymbol{U}_1, 1),$$

$$\text{and} \qquad \boldsymbol{Z}_{12} \sim N_{m,u-1}(\boldsymbol{\gamma}'\boldsymbol{U}_{12}, \boldsymbol{U}_0 - \boldsymbol{U}_1, \boldsymbol{I}_{u-1}).$$

We see that the above model is exactly same as Arnold's (1979) EGLM.

Theorem 2. *The estimates of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\gamma}$ *in the model (3) are given by*

$$\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} = (uv)^{-1/2}\Big(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\gamma}\Big), \tag{7}$$

$$\underset{r-1\times m}{\widehat{\gamma}} = (\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})^{-1}(\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11}), \tag{8}$$

Proof: The model (3) can be written as

$$\begin{aligned}\underset{uv\times m}{\boldsymbol{Y}'} &= \mathbf{1}_{uv}\underset{1\times m}{\boldsymbol{\alpha}'} + \underset{uv\times r-1}{\boldsymbol{T}'}\underset{r-1\times m}{\gamma} + \underset{uv\times m}{\boldsymbol{e}'}, \\ &= \begin{bmatrix} \mathbf{1}_{uv} & \boldsymbol{T}' \end{bmatrix}\begin{bmatrix} \boldsymbol{\alpha}' \\ \gamma \end{bmatrix} + \boldsymbol{e}', \\ &= \underset{uv\times r}{\boldsymbol{X}}\underset{r\times m}{\boldsymbol{B}} + \boldsymbol{e}'.\end{aligned} \tag{9}$$

Therefore, the least square estimate of $\boldsymbol{B}$ is given by

$$(\boldsymbol{X}'\boldsymbol{X})\underset{r\times m}{\widehat{\boldsymbol{B}}} = \boldsymbol{X}'\boldsymbol{Y}'.$$

Now,

$$\begin{aligned}\underset{r\times r}{\boldsymbol{X}'\boldsymbol{X}} &= \begin{bmatrix} \mathbf{1}'_{uv} \\ \boldsymbol{T} \end{bmatrix}\begin{bmatrix} \mathbf{1}_{uv} & \boldsymbol{T}' \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}'_{uv}\mathbf{1}_{uv} & \mathbf{1}'_{uv}\boldsymbol{T}' \\ \boldsymbol{T}\mathbf{1}_{uv} & \boldsymbol{T}\boldsymbol{T}' \end{bmatrix}\end{aligned}$$

and

$$\underset{r\times m}{\underset{r\times uv}{\boldsymbol{X}'}\underset{uv\times m}{\boldsymbol{Y}'}} = \begin{bmatrix} \mathbf{1}'_{uv}\underset{uv\times m}{\boldsymbol{Y}'} \\ \underset{r-1\times uv}{\boldsymbol{T}}\underset{uv\times m}{\boldsymbol{Y}'} \end{bmatrix}$$

We will now compute each of the partitioned matrices in $\underset{r\times r}{\boldsymbol{X}'\boldsymbol{X}}$ and $\underset{r\times m}{\underset{r\times uv}{\boldsymbol{X}'}\underset{uv\times m}{\boldsymbol{Y}'}}$. Using Lemmas 3 and 5 we have

$$\begin{aligned} & \underset{r-1\times uv}{\boldsymbol{T}}\mathbf{1}_{uv} \\ =\ & \underset{r-1\times uv}{\boldsymbol{T}}(\underset{v\times v}{\boldsymbol{C}}\otimes\boldsymbol{I}_u)(\boldsymbol{I}_v\otimes\underset{u\times u}{\boldsymbol{C}^*})(\boldsymbol{I}_v\otimes\underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes\boldsymbol{I}_u)\mathbf{1}_{uv} \\ =\ & (\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \ldots : \underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{r-1\times u-1}{\boldsymbol{U}_{v2}})(\boldsymbol{I}_v\otimes\underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes\boldsymbol{I}_u)\mathbf{1}_{uv} \\ =\ & (\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \ldots : \underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{r-1\times u-1}{\boldsymbol{U}_{v2}})\begin{bmatrix} \sqrt{uv} \\ \mathbf{0}_{u-1} \\ \mathbf{0}_u \\ \vdots \\ \mathbf{0}_u \end{bmatrix} \\ =\ & \sqrt{uv}\,\underset{r-1\times 1}{\boldsymbol{U}_{11}},\end{aligned}$$

and using Lemma 3 we have

$$
\begin{aligned}
& \underset{r-1\times uv}{\boldsymbol{T}}\underset{uv\times r-1}{\boldsymbol{T}'} \\
= \;& \underset{r-1\times uv}{\boldsymbol{T}}(\underset{v\times v}{\boldsymbol{C}}\otimes \boldsymbol{I}_u)(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^*})(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_u)\underset{uv\times r-1}{\boldsymbol{T}'} \\
= \;& (\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \ldots : \underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{r-1\times u-1}{\boldsymbol{U}_{v2}}) \begin{bmatrix} \underset{1\times r-1}{\boldsymbol{U}'_{11}} \\ \underset{u-1\times r-1}{\boldsymbol{U}'_{12}} \\ \vdots \\ \underset{1\times r-1}{\boldsymbol{U}'_{v1}} \\ \underset{u-1\times r-1}{\boldsymbol{U}'_{v2}} \end{bmatrix} \\
= \;& \sum_{i=1}^{v}\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2} \\
= \;& \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}.
\end{aligned}
$$

Now, using Lemmas 2 and 5 we have

$$
\begin{aligned}
& \mathbf{1}'_{uv}\underset{uv\times m}{\boldsymbol{Y}'} \\
= \;& \mathbf{1}'_{uv}(\underset{v\times v}{\boldsymbol{C}}\otimes \boldsymbol{I}_u)(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^*})(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_u)\underset{uv\times m}{\boldsymbol{Y}'} \\
= \;& \left[\;\sqrt{uv} : \;\; \mathbf{0}_{u-1} : \;\; \mathbf{0}_u : \;\; \cdots : \;\; \mathbf{0}_u\;\right](\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_u)\underset{uv\times m}{\boldsymbol{Y}'} \\
= \;& \sqrt{uv}\underset{1\times m}{\boldsymbol{Z}'_{11}}.
\end{aligned}
$$

Again using Lemmas 2 and 3 we get

$$
\begin{aligned}
\underset{r-1\times uv}{\boldsymbol{T}}\underset{uv\times m}{\boldsymbol{Y}'} \;=\;& \underset{r-1\times uv}{\boldsymbol{T}}(\underset{v\times v}{\boldsymbol{C}}\otimes \boldsymbol{I}_u)(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^*})(\boldsymbol{I}_v\otimes \underset{u\times u}{\boldsymbol{C}^{*\prime}})(\underset{v\times v}{\boldsymbol{C}'}\otimes \boldsymbol{I}_u)\boldsymbol{Y}' \\
=\;& (\underset{r-1\times 1}{\boldsymbol{U}_{11}} : \underset{r-1\times u-1}{\boldsymbol{U}_{12}} : \ldots : \underset{r-1\times 1}{\boldsymbol{U}_{v1}} : \underset{r-1\times u-1}{\boldsymbol{U}_{v2}}) \begin{bmatrix} \underset{1\times m}{\boldsymbol{Z}'_{11}} \\ \underset{u-1\times m}{\boldsymbol{Z}'_{12}} \\ \vdots \\ \underset{1\times m}{\boldsymbol{Z}'_{v1}} \\ \underset{u-1\times m}{\boldsymbol{Z}'_{v2}} \end{bmatrix} \\
=\;& \sum_{i=1}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2} \\
=\;& \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij}.
\end{aligned}
$$

Therefore,

$$(\boldsymbol{X}'\boldsymbol{X})\widehat{\boldsymbol{B}} = \begin{bmatrix} uv & \sqrt{uv}\,\underset{1\times r-1}{\boldsymbol{U}'_{11}} \\ \sqrt{uv}\,\underset{r-1\times 1}{\boldsymbol{U}_{11}} & \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} \end{bmatrix} \begin{bmatrix} \underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} \\ \underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} \end{bmatrix},$$

and this is equal to $\boldsymbol{X}'\boldsymbol{Y}'$, which can be expressed as follows.

$$\underset{r\times m}{\underset{r\times uv}{\boldsymbol{X}'}\underset{uv\times m}{\boldsymbol{Y}'}} = \begin{bmatrix} \sqrt{uv}\underset{1\times m}{\boldsymbol{Z}'_{11}} \\ \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} \end{bmatrix}.$$

Therefore, equating the corresponding terms we get

$$uv\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} + \sqrt{uv}\,\underset{1\times r-1}{\boldsymbol{U}'_{11}}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sqrt{uv}\underset{1\times m}{\boldsymbol{Z}'_{11}} \tag{10a}$$

$$\sqrt{uv}\,\underset{r-1\times 1}{\boldsymbol{U}_{11}}\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} + \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij}. \tag{10b}$$

Now, from (10a) we have

$$\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} = (uv)^{-1/2}\Big(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}\Big).$$

Thus, (7) is proved. Now, substituting the value of $\widehat{\boldsymbol{\alpha}}'$ in (10b) we get,

$$\sqrt{uv}\,\underset{r-1\times 1}{\boldsymbol{U}_{11}}\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} + \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij}$$

$$\sqrt{uv}\,\underset{r-1\times 1}{\boldsymbol{U}_{11}}(uv)^{-1/2}\Big(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}\Big) + \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij}$$

$$\underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}} + \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij}$$

$$(\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11}$$

Therefore,

$$\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = (\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})^{-1}(\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11})$$

If $v=1$, the model reduces to Arnold's (1979) EGLM. For $v=1$ we see that

$$\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} = (u)^{-1/2}\Big(\boldsymbol{Z}'_{11} - \boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}\Big),$$

$$\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} = (\sum_{j=1}^{2}\boldsymbol{U}_{1j}\boldsymbol{U}'_{1j} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})^{-1}(\sum_{j=1}^{2}\boldsymbol{U}_{1j}\boldsymbol{Z}'_{1j} - \underset{r-1\times 1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11})$$

$$= (\underset{r-1\times 1}{\boldsymbol{U}_{12}}\underset{1\times r-1}{\boldsymbol{U}'_{12}})^{-1}(\underset{r-1\times u-1}{\boldsymbol{U}_{12}}\underset{u-1\times m}{\boldsymbol{Z}'_{12}}),$$

and these estimates are exactly same as obtained by Arnold (1979)

Theorem 3. *The distributions of* $\widehat{\boldsymbol{\alpha}}'$ *and* $\widehat{\boldsymbol{\gamma}}$ *are as follows:*

$$\begin{aligned}\widehat{\boldsymbol{\alpha}}' &\sim N_{1,m}\Big(\boldsymbol{\alpha}', 1, \\ &\frac{1}{uv}\Big[\boldsymbol{\Delta}_3 + \boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2 \\ &+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big),\end{aligned}$$

$$\begin{aligned}\widehat{\boldsymbol{\gamma}} &= \boldsymbol{A}^{-1}\Big(\sum_{i=1}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}\Big) \sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, \\ &\boldsymbol{A}^{-1}\sum_{i=1}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big).\end{aligned}$$

where $\underset{r-1\times r-1}{\boldsymbol{A}} = (\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \underset{r-1\times1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})$.

Proof: We will first find the distribution of $\widehat{\boldsymbol{\gamma}}$, and then the distribution of $\widehat{\boldsymbol{\alpha}}'$. We note that $\boldsymbol{A} = \boldsymbol{A}'$, so $\boldsymbol{A}$ is symmetric. Now, from (8) we have

$$\begin{aligned}\underset{r-1\times m}{\widehat{\boldsymbol{\gamma}}} &= (\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \underset{r-1\times1}{\boldsymbol{U}_{11}}\boldsymbol{U}'_{11})^{-1}(\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} - \underset{r-1\times1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11}) \\ &= \boldsymbol{A}^{-1}(\sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{Z}'_{ij} - \underset{r-1\times1}{\boldsymbol{U}_{11}}\boldsymbol{Z}'_{11}) \\ &= \boldsymbol{A}^{-1}[\boldsymbol{U}_{21}\boldsymbol{Z}'_{21} + \cdots + \boldsymbol{U}_{v1}\boldsymbol{Z}'_{v1} + \boldsymbol{U}_{12}\boldsymbol{Z}'_{12} + \boldsymbol{U}_{22}\boldsymbol{Z}'_{22} + \cdots + \boldsymbol{U}_{v2}\boldsymbol{Z}'_{v2}] \\ &= \boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}\end{aligned}$$

To get the distribution of $\widehat{\boldsymbol{\gamma}}$ we will find the distributions of $\boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1}$ and $\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}$ separately. Using (6b) and (6c) we have

$$\begin{aligned}\widehat{\boldsymbol{\gamma}}_1 = \boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} &\sim N_{r-1,m}(\boldsymbol{A}^{-1}\sum_{i=2}^{v}(\boldsymbol{U}_{i1})(\boldsymbol{U}'_{i1}\boldsymbol{\gamma}), \\ &\qquad \boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}, \boldsymbol{\Delta}_2),\end{aligned}$$

and,

$$\widehat{\boldsymbol{\gamma}}_2 = \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2} \sim N_{r-1,m}(\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma}, \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}, \boldsymbol{\Delta}_1).$$

Therefore, We have

$$\widehat{\boldsymbol{\gamma}}_1 = \boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} \sim N_{(r-1),m}(\boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1}\boldsymbol{\gamma},$$
$$(\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2),$$

and

$$\widehat{\boldsymbol{\gamma}}_2 = \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2} \sim N_{(r-1),m}(\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma}, \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1})\otimes\boldsymbol{\Delta}_1.$$

Therefore,

$$\begin{aligned}
\widehat{\boldsymbol{\gamma}} &= \boldsymbol{A}^{-1}\Big(\sum_{i=2}^{v}(\boldsymbol{U}_{i1})\boldsymbol{Z}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}\Big)\\
&= \widehat{\boldsymbol{\gamma}}_1 + \widehat{\boldsymbol{\gamma}}_2\\
&\sim N_{(r-1),m}\Big(\boldsymbol{A}^{-1}\sum_{i=2}^{v}(\boldsymbol{U}_{i1})\boldsymbol{U}'_{i1}\boldsymbol{\gamma} + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma},\\
&\qquad \boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
\boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1}\boldsymbol{\gamma} + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma} &= \boldsymbol{A}^{-1}[\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}]\boldsymbol{\gamma}\\
&= \boldsymbol{\gamma}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{\boldsymbol{\gamma}} &= \boldsymbol{A}^{-1}\Big(\sum_{i=1}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}\Big) \sim N_{(r-1),m}\Big(\boldsymbol{\gamma},\\
&\qquad \boldsymbol{A}^{-1}\sum_{i=1}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big).
\end{aligned}$$

We see that $\widehat{\boldsymbol{\gamma}}$ is an unbiased estimate of $\boldsymbol{\gamma}$. We will now find the distribution of $\widehat{\boldsymbol{\alpha}}'$. Now from (7) we have

$$\begin{aligned}
\underset{1\times m}{\widehat{\boldsymbol{\alpha}}'} &= \frac{1}{\sqrt{uv}}\boldsymbol{Z}'_{11} - \frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}},\\
\widehat{\boldsymbol{\alpha}}' &= \frac{1}{\sqrt{uv}}\boldsymbol{Z}'_{11} - \frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}(\widehat{\boldsymbol{\gamma}_1} + \widehat{\boldsymbol{\gamma}_2}),\\
&= \frac{1}{\sqrt{uv}}\boldsymbol{Z}'_{11} - \frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}_1 - \frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}_2.
\end{aligned} \tag{11}$$

Now, from (6a) we have

$$\begin{aligned}
\frac{1}{\sqrt{uv}}\boldsymbol{Z}'_{11} &\sim N_{1,m}\Big(\frac{1}{\sqrt{uv}}(\sqrt{uv}\boldsymbol{\alpha}'+\boldsymbol{U}'_{11}\boldsymbol{\gamma}),1,\frac{1}{uv}\boldsymbol{\Delta}_3\Big),\\
\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}_1 &\sim N_{1,m}\Big(\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}(\boldsymbol{U}_{i1})(\boldsymbol{U}'_{i1}\boldsymbol{\gamma}),1,\\
&\qquad \frac{1}{uv}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2\Big),\\
\text{and,}\quad \frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\widehat{\boldsymbol{\gamma}}_2 &\sim N_{1,m}\Big(\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma},1,\\
&\qquad \frac{1}{uv}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{1}{\sqrt{uv}}(\sqrt{uv}\boldsymbol{\alpha}'+\boldsymbol{U}'_{11}\boldsymbol{\gamma})-\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}(\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1}\boldsymbol{\gamma})-\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma}\\
=\;&\boldsymbol{\alpha}'+\frac{1}{\sqrt{uv}}\boldsymbol{U}'_{11}\left[\boldsymbol{\gamma}-\boldsymbol{A}^{-1}\sum_{i=2}^{v}\boldsymbol{U}_{i1}\boldsymbol{U}'_{i1}\boldsymbol{\gamma}-\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{\gamma}\right]\\
=\;&\boldsymbol{\alpha}'
\end{aligned}$$

Therefore, from (11) we have

$$\begin{aligned}
\widehat{\boldsymbol{\alpha}}' \sim\; & N_{1,m}\Big(\boldsymbol{\alpha}',1,\\
&\frac{1}{uv}\Big[\boldsymbol{\Delta}_3+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2\\
&+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big).
\end{aligned}$$

Here also we see that $\widehat{\boldsymbol{\alpha}}$ is an unbiased estimate of $\boldsymbol{\alpha}$.

Corollary 4. *If* $\mathbf{W}=\mathbf{0}$*, the distributions of* $\widehat{\boldsymbol{\alpha}}'$ *and* $\widehat{\boldsymbol{\gamma}}$ *are as follows:*

$$\begin{aligned}
\widehat{\boldsymbol{\alpha}}' \sim\; & N_{1,m}\Big(\boldsymbol{\alpha}',1,\\
&\frac{1}{uv}\Big[\boldsymbol{\Delta}_2+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2\\
&+\boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big),
\end{aligned}$$

and

$$\begin{aligned}\widehat{\boldsymbol{\gamma}} &= \boldsymbol{A}^{-1}\Big(\sum_{i=1}^{v}\boldsymbol{U}_{i1}\boldsymbol{Z}'_{i1} + \sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{Z}'_{i2}\Big) \sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, \\ & \boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big).\end{aligned}$$

where $\underset{r-1\times r-1}{\boldsymbol{A}} = \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11}$.

Corollary 5. *If* $v = 1$*,*

$$\begin{aligned}\underset{r-1\times r-1}{\boldsymbol{A}} &= \sum_{j=1}^{2}\sum_{i=1}^{v}\boldsymbol{U}_{ij}\boldsymbol{U}'_{ij} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11} \\ &= \boldsymbol{U}_{11}\boldsymbol{U}'_{11} + \boldsymbol{U}_{12}\boldsymbol{U}'_{12} - \boldsymbol{U}_{11}\boldsymbol{U}'_{11} \\ &= \boldsymbol{U}_{12}\boldsymbol{U}'_{12},\end{aligned}$$

and the distributions of $\widehat{\boldsymbol{\alpha}}'$ *and* $\widehat{\boldsymbol{\gamma}}$ *are as follows:*

$$\begin{aligned}\widehat{\boldsymbol{\alpha}}' &\sim N_{1,m}\Big(\boldsymbol{\alpha}', 1, \\ &\quad \frac{1}{uv}\Big[\boldsymbol{\Delta}_3 + \boldsymbol{U}'_{11}\boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_2 \\ &\quad \frac{1}{u}\Big[((\boldsymbol{U}_0 - \boldsymbol{U}_1) + u\boldsymbol{U}_1) + \boldsymbol{U}'_{11}(\boldsymbol{U}_{12}\boldsymbol{U}'_{12})^{-1}\boldsymbol{U}_{11}\boldsymbol{\Delta}_1\Big]\Big]\Big) \\ &\sim N_{1,m}\Big(\boldsymbol{\alpha}', 1, \boldsymbol{U}_1 \\ &\quad + \frac{1}{u}\left(1 + \boldsymbol{U}'_{11}(\boldsymbol{U}_{12}\boldsymbol{U}'_{12})^{-1}\boldsymbol{U}_{11}\right)(\boldsymbol{U}_0 - \boldsymbol{U}_1)\Big)\end{aligned}$$

and

$$\begin{aligned}\widehat{\boldsymbol{\gamma}} &\sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, \boldsymbol{A}^{-1}\sum_{i=2}^{v}\left[(\boldsymbol{U}_{i1})(\boldsymbol{U}_{i1})'\right]\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_2 + \boldsymbol{A}^{-1}\sum_{i=1}^{v}\boldsymbol{U}_{i2}\boldsymbol{U}'_{i2}\boldsymbol{A}^{-1}\otimes\boldsymbol{\Delta}_1\Big) \\ &\sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, (\boldsymbol{U}_{12}\boldsymbol{U}'_{12})^{-1}\boldsymbol{U}_{12}\boldsymbol{U}'_{12}(\boldsymbol{U}_{12}\boldsymbol{U}'_{12})^{-1}\otimes\boldsymbol{\Delta}_1\Big) \\ &\sim N_{(r-1),m}\Big(\boldsymbol{\gamma}, (\boldsymbol{U}_{12}\boldsymbol{U}'_{12})^{-1}, (\boldsymbol{U}_0 - \boldsymbol{U}_1)\Big)\end{aligned}$$

These estimates are exactly the same as those of Arnold's (1979).

Acknowledgements

The author would like to acknowledge the generous support for the summer grant from the College of Business at the University of Texas at San Antonio.

References

[1] Anderson, T.W. An introduction to multivariate statistical analysis, Third Ed. Wiley, New Jersey, 2003.

[2] Arnold, S.F., Linear Models with Edchangeably Distributed Errors. Journal of the American Statistical Association 74 (365) (1979) 194-199.

[3] Gupta, A.K., Nagar, D.K. Matrix Variate Distributions, Chapman & Hall/CRC, Florida, 2000.

[4] Leiva, R., Linear Discrimination with Equicorrelated Training Vectors, J. Multivariate Analysis, 98(2) (2007) 384-409.

[5] Pan, J.X., Fang, K.T. Growth Curva Models and Statistical Diagnostics, Springer-Verlag, New York, 2002.

[6] Roy, A., Leiva, R. Discrimination with jointly equicorrelated multi-level multivariate data. Advances in Data Analysis and Classification, 1(3), (2007) 175-199.