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Abstract

Many real life decision making problems can be modeled as stochastic multi-attribute decision making (MADM) problems. A novel method for stochastic MADM problems is developed based on the ideal and nadir solutions as in the classical TOPSIS method. In a stochastic MADM problem, the evaluations of the alternatives with respect to the different attributes are represented by discrete stochastic variables. According to stochastic dominance rules, the probability distributions of the ideal and nadir variates, both are discrete stochastic variables, are defined and determined for a set of stochastic variables. A metric is proposed to measure the distance between two discrete stochastic variables. The ideal solution is a vector of ideal variates and the nadir solution is vector of nadir variates for the multiple attributes. As in the classical TOPSIS method, the relative closeness of an alternative is determined by its distances from the ideal and nadir solutions. The rankings of the alternatives are determined using the relative closeness. Examples are presented to illustrate the effectiveness of the proposed method. Through the examples, several significant advantages of the proposed method over some existing methods are discussed.

Keywords: stochastic MADM, TOPSIS, ranking, ideal and nadir solutions, relative closeness

JEL Classification Codes: C61, C63, C64
ACM Classification Codes: F.2.3, G.1.6

1. Introduction

Multi-attribute decision making (MADM) methods have been applied to a wide range of real-world problems. Most of these methods appeared in the literature focus on cases where attribute values are crisp numbers or fuzzy numbers (Hwang and Yoon, 1981; Fan *et al*., 2006; Jiang *et al*., 2008). MADM problems usually have stochastic attributes, *i.e*., attributes that are stochastic or random variables. Consistent with the convention in the literature of stochastic MADM, the term “stochastic variables” rather than “random variables” will be used in the following. There are many such examples in real life. In a forest site productivity evaluation problem, ecological interpretability is a stochastic variable (Chuu, 2009). In decision support for investing in a potential industry, the environment conditions are stochastic variables (Zhang *et al*., 2010). In the selection of the most desirable computer development project, the chance of success is a stochastic variable (Nowak *et al*., 2004). In the selection of a site for a waste treatment facility, transportation cost is a stochastic variable (Lahdelma *et al*., 2002). In the selection of a strategic decision support model for a retailer’s operation, market share is a stochastic variable (Sarker and Quaddus, 2002). In the formation of a management strategy for a forest ecosystem, net income from timber cuttings during the planning period is a stochastic variable (Lahdelma and Salminen, 2009). Hence, the development of MADM methods with the capacity of handling stochastic attributes has attracted the attention of many researchers.

Some approaches have been proposed to solve stochastic MADM problems from different perspectives (Fan *et al*., 2010). Keeney and Raiffa (1976) initially proposed a method based on multi-attribute utility theory (MAUT) for dealing with MADM problems under uncertainty. Martel and D’Avignon (1982) and Martel *et al*. (1986) aggregated evaluations of multiple experts to obtain random evaluations, called distributive evaluations, on the alternatives. Two indices, a confidence index and a doubt index, are calculated using these evaluations. A degree of credibility is obtained by combing the two indices. A fuzzy outranking relation characterized by the degree of credibility is used to capture the preferences of one alternative over another. D’Avignon and Vincke (1988) considered the preference indices given by the decision maker (DM), and proposed a multi-attribute procedure to aggregate random evaluations of the alternatives into random preference degrees.

Stochastic dominance (SD) rules have been used to solve the stochastic MADM problem. Martel and Zaras (1995) used SD rules and utility functions to determine the outranking relations between alternatives on each attribute. Based on these outranking relations, they used the ELECTRE method (Roy, 1985, 1991) to obtain the rankings of the alternatives. Zaras (1999) used a rough set approach for obtaining a set of decision rules. Based on these rules, the non-redundant set of attributes is identified. By applying multi-attribute SD rules to the reduced set of attributes, the rankings of the alternatives are obtained. Zaras (2001) combined the SD rules and the rough set approach to study the MADM problem with deterministic and stochastic evaluation information. Rough set is used to reduce the size of the attribute set. Multi-attribute SD rules defined by Zaras (2001) are used to determine the dominance relations of alternatives on the smaller set of attributes. Zaras (2004) studied the MADM problem with deterministic, fuzzy and stochastic evaluation information. He proposed mixed-data multi-attribute dominance to identify the preference relation between alternatives on each attribute. Based on the dominance relations, several decision rules are generated by using the rough set approach and the rankings of the alternatives are obtained subsequently. For MADM problems with stochastic information, Nowak (2004) used SD rules to determine the dominance relationship associated with a pair of alternatives, and then identified strict preference, weak preference and indifference preference between alternatives on a single attribute. The rankings of the alternatives are then obtained by using the ELECTRE-III distillation procedure (Roy, 1985 1991). Nowak (2007) studied the stochastic MADM problem using the DM's aspiration information. The number of alternatives is progressively reduced according to the DM's aspiration threshold. Furthermore, he used the SD rules to select the desirable alternative from the reduced set of alternatives. Zhang *et al.* (2010) introduced the concept of stochastic dominance degree (SDD) to measure the strength of dominance of one alternative over another. Based on the overall SDD matrix, the rankings of the alternatives are obtained by using PROMETHEE-II (Brans and Vincke, 1985; Kolli and Parsaei, 1992).

In addition, other methods for solving stochastic MADM problems have been reported in the literature. Văduva and Resteanu (2009) examined a MADM problem with stochastic attribute values. They first standardized the stochastic attribute values. The standardized stochastic attribute values are then transformed into Shannon's entropy or Onicescu's informational energy. By using a simple additive weighting approach or using TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), the Shannon's entropy or Onicescu's informational energy of each alternative is aggregated to obtain the rankings of all alternatives. Fan *et al.* (2010) proposed a method based on pairwise comparisons of alternatives with random evaluations to solve stochastic MADM problems. After computing superior, indifferent and inferior probabilities on pairwise comparisons of alternatives, the rankings of all alternatives are obtained. Fan *et al.* (2013) proposed a method based on the ideal and anti-ideal points for the stochastic MADM problem, where consequences of alternatives with respect to attributes are represented by stochastic variables with cumulative distribution functions. In this current study, the term "nadir solution" or "nadir point" instead of "anti-ideal point" is used. Ideal solutions are also called positive ideal solutions and nadir solutions are also called negative ideal solutions in the literature.

Prior studies have significantly enriched the theories and techniques of stochastic MADM problems. However, there are still limitations with existing methods. For example, in methods using MAUT, the utility function is often difficult to obtain (Nowak, 2004). In methods using confidence indices and preference indices, the meanings of these indices are sometimes not easily interpretable (Stewart, 2005). Methods based on SD rules sometimes cannot determine or identify the dominance relation between two distinct alternatives (Leshno and Levy, 2002). When the dominance relation cannot be established, the rankings of the alternatives cannot be determined. In methods using TOPSIS, the stochastic attribute values are transformed into crisp or interval values. Obviously, this transformation causes information loss. In methods using ideal and anti-ideal points, the cumulative distribution functions of discrete stochastic variables are obtained to determine the ideal and anti-ideal points. However, the probabilities of the ideal and anti-ideal points at some possible values may be different from those of the stochastic attribute values. Hence, the probability distribution functions of ideal and anti-ideal points may be independent of those of the stochastic attribute values. However, in the classical TOPSIS method, the ideal and anti-ideal points are used to measure the preference relations using the attribute values. Hence, it is valuable to develop a novel method for solving stochastic MADM problems so as to overcome these limitations.

Inspired by the use of ideal and nadir solutions in the TOPSIS method, this study provides a new way of ranking the alternatives when the attributes are discrete stochastic variables. The proposed method is motivated by the following ideas. Just like in the TOPSIS method, the chosen alternative should have the shortest distance from the ideal solution and the longest distance from the nadir solution. The method should overcome some of the limitations in the existing methods. Furthermore, this method should not need much judgmental input from the users such as the confidence indices, preference indices and/or utility functions in some existing methods, and should not need to verify the SD relations.

The developed method has three major components. Using SD relations, an ideal variate and a nadir variate for a set of stochastic variables are defined first. The ideal and nadir variates are defined on a single attribute. For stochastic MADM problems, the ideal solution is a vector of ideal variates and the nadir solution is a vector of nadir variates. A metric measuring the distance between two discrete stochastic variables is then defined. Using the ideal and nadir solutions and the metric, the method for the stochastic MADM problem is finally developed. In the method, the probability distributions of the ideal and nadir solutions are determined first, the distances of each alternative from the ideal and the nadir solutions are then calculated, and the relative closeness of each alternative is finally calculated by using these distances. The relative closeness of the alternatives is then used to obtain the rankings of the alternatives.

The rest of this paper is organized as follows. A brief introduction to the classical TOPSIS method is given in Section 2. Stochastic dominance relations, ideal and nadir variates, and stochastic expectations are discussed in Section 3. A metric measuring the distance between two discrete stochastic variables is presented in Section 4. The method based on the ideal and nadir solutions to solve the stochastic MADM problem is presented in Section 5. Two examples illustrating the feasibility and effectiveness of the proposed method are presented in Section 6. Summaries and conclusions are given in Section 7.

2. A brief introduction to the classical TOPSIS method

Originally proposed by Hwang and Yoon (1981), the TOPSIS method is an effective tool for dealing with MADM problems (Awasthi *et al*., 2011; Dymova *et al*., 2013; Khalili-Damghani *et al*., 2013; Yue, 2012). TOPSIS simultaneously considers the distances from both the ideal and the nadir solutions. The alternatives are ranked by the relative closeness combining the two distances.

Suppose a MADM problem has m alternatives and n attributes. For convenience of analysis, it is assumed that all the attributes are of benefit type, *i.e.*, a larger value is preferred to a smaller value for any attribute of any alternative. The value of alternative i on attribute j is represented by x_{ij}, for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. The decision matrix consists of all these values and is represented by $X=[x_{ij}]_{m\times n}$. The TOPSIS method has the following steps:

Step 1: Construct the normalized decision matrix $Y=[y_{ij}]_{m\times n}$ using (1) in the following

$$y_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \quad i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n. \tag{1}$$

Step 2: Calculate the weighted normalized decision matrix $F = [f_{ij}]_{m \times n}$ using (2) in the following

$$f_{ij} = w_j y_{ij}, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \tag{2}$$

where w_j is the weight assigned to attribute j by the DM with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Step 3: Determine the ideal solution $F^+ = \{f_1^+, \ldots, f_n^+\}$ and the nadir solution $F^- = \{f_1^-, \ldots, f_n^-\}$ using (3) and (4), respectively, in the following

$$f_j^+ = \max\{f_{ij} \mid i = 1, \ldots, m\}, \tag{3}$$

$$f_j^- = \min\{f_{ij} \mid i = 1, \ldots, m\}. \tag{4}$$

Step 4: Calculate the distances of each alternative from the ideal and nadir solutions. The distance of alternative i from the ideal solution is defined in (5) in the following

$$d_i^+ = \sqrt{\sum_{j=1}^{n} (f_{ij} - f_j^+)^2}, \quad i = 1, 2, \ldots, m. \tag{5}$$

Similarly, the distance of alternative i from the nadir solution is defined in (6) in the following

$$d_i^- = \sqrt{\sum_{j=1}^{n} (f_{ij} - f_j^-)^2}, \quad i = 1, 2, \ldots, m. \tag{6}$$

Step 5: Calculate the relative closeness c_i of alternative i to the ideal and nadir solutions using (7) in the following

$$c_i = \frac{d_i^-}{d_i^- + d_i^+}, \quad i = 1, 2, \ldots, m. \tag{7}$$

Step 6: Obtain the rankings of the alternatives according to the relative closeness c_i. The larger c_i is, the higher alternative i is ranked.

As seen from the above steps, the basic idea of TOPSIS is rather straightforward. A good property of the TOPSIS method is that the chosen alternative has the shortest distance from the ideal solution and the longest distance from the nadir solution. Its simple computation process can be easily programmed into a spreadsheet (Shih *et al.*, 2007). However, when the TOPSIS method is applied to the stochastic MADM problem, some steps need to be modified, especially when working with stochastic attribute values with different scales. These modifications include the normalization of the stochastic attribute values, the identification of the ideal and nadir solutions among a group of stochastic attribute values and the calculation of the distances of each alternative from the ideal and nadir solutions. All the modifications are discussed in this paper.

3. Ideal and nadir variates and their properties

The attributes are discrete stochastic variables in the stochastic MADM problem concerned in this study. A discrete stochastic variable, represented by ξ, is a variable that has a finite number, represented by l, of possible positive values. Let X^+ be the domain, *i.e.*, the set of all the possible positive values that ξ may take. The elements of X^+ are arranged in ascending order, *i.e.*, $X^+ = \{x_1, x_2, \ldots, x_l\}$ with $x_1 < x_2 < \cdots < x_l$ and $x_1, x_2, \ldots, x_l > 0$. Let $P_v(\xi)$ be the probability that ξ is equal to x_v, with $P_v(\xi) \geq 0$, for any $x_v \in X^+$ and $v = 1, 2, \ldots, l$, and $\sum_{v=1}^{l} P_v(\xi) = 1$. If $P_v(\xi) = 1$ for any $v = 1, 2, \ldots, l$, the discrete stochastic variable ξ becomes deterministic, *i.e.*, degenerates to a real number x_v.

In the following, SD concepts and definitions are discussed first. After the ideal and nadir variates are defined, their properties are then presented. These properties are stated in SD relations and stochastic expectations.

3.1 Stochastic dominance relations

SD relations (Martel and Zaras, 1995; Nowak, 2004) are often used to compare stochastic prospects. A stochastic prospect is a probabilistic alternative with known probabilities for its outcomes. SD relations include first order stochastic dominance (FSD), second order stochastic dominance (SSD) and third order stochastic dominance (TSD). These SD relations are defined on their probability distributions in the following.

Definition 1. Let ξ_1 and ξ_2 be any two discrete stochastic variables, and let $F_1(x)$ and $F_2(x)$ be the cumulative distribution functions of ξ_1 and ξ_2, respectively. Then for $x_1, x_l \in X^+$ with $x_1 \leq x_l$, the SD relations are

(1) $F_1(x)$ FSD $F_2(x)$ if and only if $F_1(x) \neq F_2(x)$ and $F_1(x) - F_2(x) \leq 0$ for all $x_1 \leq x \leq x_l$.

(2) $F_1(x)$ SSD $F_2(x)$ if and only if $F_1(x) \neq F_2(x)$ and $\int_{x_1}^{x} (F_1(y) - F_2(y))dy \leq 0$ for all $x_1 \leq x \leq x_l$.

(3) $F_1(x)$ TSD $F_2(x)$ if and only if $F_1(x) \neq F_2(x)$ and $\int_{x_1}^{x}\int_{x_1}^{z} (F_1(y) - F_2(y))dydz \leq 0$ for all $x_1 \leq x \leq x_l$.

Remark 1. If $F_1(x)$ FSD $F_2(x)$, then $F_1(x)$ SSD $F_2(x)$ and $F_1(x)$ TSD $F_2(x)$; if $F_1(x)$ SSD $F_2(x)$, then $F_1(x)$ TSD $F_2(x)$.

Remark 2. Let ξ_1, ξ_2 and ξ_3 be any three discrete stochastic variables, and $F_1(x)$, $F_2(x)$ and $F_3(x)$ be the cumulative distribution functions of ξ_1, ξ_2 and ξ_3, respectively. For $x_1, x_l \in X^+$ with $x_1 \leq x_l$, if $F_1(x)$ FSD $F_2(x)$, and $F_2(x)$ FSD $F_3(x)$, then $F_1(x)$ FSD $F_3(x)$.

From $F_1(x) - F_2(x) \leq 0$, we have $1 - F_2(x) - (1 - F_1(x)) \leq 0$, where

$$1-F_2(x)-(1-F_1(x))=\begin{cases}0, & x \ge x_l \\ P_l(\xi_2)-P_l(\xi_1), & x_{l-1} \le x < x_l \\ \sum_{v=l-1}^{l} P_v(\xi_2)-\sum_{v=l-1}^{l} P_v(\xi_1), & x_{l-2} \le x < x_{l-1} \\ \vdots & \vdots \\ \sum_{v=3}^{l} P_v(\xi_2)-\sum_{v=3}^{l} P_v(\xi_1), & x_2 \le x < x_3 \\ \sum_{v=2}^{l} P_v(\xi_2)-\sum_{v=2}^{l} P_v(\xi_1), & x_1 \le x < x_2 \\ 0, & x < x_1 \end{cases}.$$

3.2 Definitions of the ideal and nadir variates

In a stochastic MADM problem, the ideal and nadir solutions are vectors of discrete stochastic variables. Each component of the ideal solution is an ideal variate and each component of the nadir solution is a nadir variate. Each variate is defined by a set of discrete values with an associated probability distribution. For the evaluation of an alternative in a discrete stochastic MADM problem, larger probabilities at larger possible values often indicate a better evaluation of this alternative. Conversely, larger probabilities at smaller possible values indicate a worse evaluation of this alternative. Like the ideal and nadir solutions in the classical TOPSIS method, the ideal and nadir solutions correspond to the possible best and worst evaluations, respectively. Thus, the ideal variate defined in this study has the maximum possible probabilities among a set of discrete stochastic variables at larger possible values. Likewise, the nadir variate has the maximum possible probabilities among a set of discrete stochastic variables at smaller possible values. Consequently, also from Definition 1 and Remarks 1 and 2, the probability distributions of the ideal and nadir variates are defined in the following.

Definition 2. Let ξ_1, ξ_2, $\cdots,\xi_h$ be h discrete stochastic variables with $h \ge 2$. The ideal variate ξ^+ has a probability distribution $P(\xi^+)$ as defined in (8) in the following

$$P(\xi^+)=\begin{cases}P_1(\xi^+)=0, & \xi^+=x_1 \\ \vdots & \vdots \\ P_{k-1}(\xi^+)=0, & \xi^+=x_{k-1} \\ P_k(\xi^+)=1-\sum_{v=k+1}^{l} P_v(\xi^+), & \xi^+=x_k \\ P_{k+1}(\xi^+)=\max_u\{P_{k+1}(\xi_u)\,|\,u=1,2,\ldots,h\}, & \xi^+=x_{k+1} \\ \vdots & \vdots \\ P_l(\xi^+)=\max_u\{P_l(\xi_u)\,|\,u=1,2,\ldots,h\}, & \xi^+=x_l \end{cases}, \tag{8}$$

such that $\sum_{v=1}^{l} P_v(\xi^+)=1$.

In this definition of $P(\xi^+)$ in (8), the subscript k is the smallest subscript such that $P_k(\xi^+) \ge 0$, where k is determined in such a way that $P_v(\xi^+)=0$ for all $1 \le v \le k-1$, $P_k(\xi^+)=1-\sum_{v=k+1}^{l} P_v(\xi^+)$, and $P_v(\xi^+)=\max_u\{P_v(\xi_u)\,|\,u=1,2,\ldots,h\}$ for all $k+1 \le v \le l$. As a result, $\sum_{v=k+1}^{l} P_v(\xi^+)=1$ is realized. In (8), determining k is an important step for constructing the probability distribution of the ideal variate ξ^+. If

$\sum_{v=q}^{l} \max_u \{P_v(\xi_u) \mid u = 1,2,\ldots,h\} > 1$ and $\sum_{v=q+1}^{l} \max_u \{P_v(\xi_u) \mid u = 1,2,\ldots,h\} < 1$ for $q \in \{1,2,\ldots,l-1\}$, then $k = q$ and $P_q(\xi^+) = 1 - \sum_{v=q+1}^{l} P_v(\xi^+)$. The discussion above also outlines a procedure to find the value of k.

Definition 3. Let ξ_1, ξ_2, $\cdots, \xi_h$ be h discrete stochastic variables with $h \geq 2$. The nadir variate ξ^- has a probability distribution $P(\xi^-)$ as defined in (9) in the following

$$P(\xi^-) = \begin{cases} P_1(\xi^-) = \max_u \{P_1(\xi_u) \mid u = 1,2,\ldots,h\}, & \xi^- = x_1 \\ \vdots & \vdots \\ P_{t-1}(\xi^-) = \max_u \{P_{t-1}(\xi_u) \mid u = 1,2,\ldots,h\}, & \xi^- = x_{t-1} \\ P_t(\xi^-) = 1 - \sum_{v=1}^{t-1} P_v(\xi^-), & \xi^- = x_t \\ P_{t+1}(\xi^-) = 0, & \xi^- = x_{t+1} \\ \vdots & \vdots \\ P_l(\xi^-) = 0, & \xi^- = x_l \end{cases}, \tag{9}$$

such that $\sum_{v=1}^{l} P_v(\xi^-) = 1$.

Similar to k in the definition of ξ^+ in (8), the subscript t in the definition of ξ^- in (9) is the largest subscript such that $P_t(\xi^-) \geq 0$, where t is determined in such a way that $P_v(\xi^-) = \max_u \{P_v(\xi_u) \mid u = 1,2,\ldots,h\}$ for all $1 \leq v \leq t-1$, $P_t(\xi^-) = 1 - \sum_{v=1}^{t-1} P_v(\xi^-)$, and $P_{t+1}(\xi^-) = 0$ for all $t+1 \leq v \leq l$. As a result, $\sum_{v=1}^{l} P_v(\xi^-) = 1$ is realized. In (9), determining t is also an important step for constructing the probability distribution of the nadir variate ξ^-. If $\sum_{v=1}^{q-1} \max_u \{P_v(\xi_u) \mid u = 1,2,\ldots,h\} < 1$ and $\sum_{v=1}^{q} \max_u \{P_v(\xi_u) \mid u = 1,2,\ldots,h\} > 1$ for $q \in \{2,\ldots,l\}$, then $t = q$ and $P_q(\xi^-) = 1 - \sum_{v=1}^{q-1} P_v(\xi^-)$. The discussion above also outlines a procedure to find the value for t. Note that the value of t is independent of that of k.

Table 1. Probability distributions of ξ_1, ξ_2 and ξ_3 as well as of ξ^+ and ξ^-

X^+	ξ_1	ξ_2	ξ_3	ξ^+	ξ^-
1		2/7			2/7
2		2/7	3/7		3/7
3	2/7	3/7		1/7	2/7
4	2/7		3/7	3/7	
5	3/7		1/7	3/7	

Example 1. The domain X^+ and the probability distributions of $h = 3$ discrete stochastic variables ξ_1, ξ_2 and ξ_3 are presented in Table 1. Each discrete stochastic variable may take on $l = 5$ possible positive values. From the probability distributions of these $h = 3$ discrete stochastic variables, the probability distributions of the ideal and nadir variates ξ^+ and ξ^- are determined. The results are also presented in Table 1. From the probability distributions of ξ^+ and ξ^-, $k = 3$ in (8) and $t = 3$ in (9) are found for this

example. The graphical exposition of the probability distributions of ξ_1, ξ_2 and ξ_3 as well as those of ξ^+ and ξ^- are given in Figure 1.

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Figure 1. appears about here

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3.3 Properties of the ideal and nadir variates

Let $F_+(x)$ and $F_-(x)$ be the cumulative distribution functions of ξ^+ and ξ^-, respectively. Let $F_u(x)$ be the cumulative distribution function of ξ_u for $u=1,2,\ldots,h$. The properties of ξ^+ and ξ^- are stated as the following proposition and corollary.

Proposition 1. (1) $F_+(x)$ FSD $F_u(x)$, $F_+(x)$ SSD $F_u(x)$, and $F_+(x)$ TSD $F_u(x)$, for $u=1,2,\ldots,h$.

(2) $F_u(x)$ FSD $F_-(x)$, $F_u(x)$ SSD $F_-(x)$, and $F_u(x)$ TSD $F_-(x)$, for $u=1,2,\ldots,h$.

The proof of Proposition 1 is provided in the Appendix. The following conclusion is directly from Proposition 1 and Remark 2.

Corollary 1. $F_+(x)$ FSD $F_-(x)$, $F_+(x)$ SSD $F_-(x)$, and $F_+(x)$ TSD $F_-(x)$.

3.4 Stochastic expectations

Stochastic expectations (Bawa, 1975; Hadar and Russell, 1969), *i.e.*, the expectations of stochastic variables, are also often used to compare stochastic prospects. They are used to show the rationales of the ideal and nadir variates in this study. The stochastic expectation of a discrete stochastic variable is defined in the following.

Definition 4. The expectation of a discrete stochastic variable ξ_u, $E(\xi_u)$, is given by

$$E(\xi_u)=\sum_{v=1}^{l}x_v P_v(\xi_u)\,. \tag{10}$$

The following proposition shows another property of the ideal and nadir variates. This property explains the rationales of the ideal and nadir variates through stochastic expectations.

Proposition 2. For any h discrete stochastic variables ξ_u for $u=1,2,\ldots,h$ and $h\geq 2$ with ξ^+ and ξ^- defined above, the expectations of ξ^+ and ξ^- satisfy $E(\xi^+)\geq E(\xi_u)$ and $E(\xi_u)\geq E(\xi^-)$.

The proof of Proposition 2 is given in the Appendix.

4. A metric measuring the distance between two discrete stochastic variables

The distance between two discrete stochastic variables is determined by two factors, *i.e.*, their values and their probability distributions. The distance is small between two discrete stochastic variables with similar values and similar probability distributions. On the other hand, the distance is large between two stochastic variables with different values and different probability distributions. In fact, several different metrics measuring the distance between discrete stochastic variables have been proposed although they may have some limitations. For example, some scholars suggested using the difference between the expectations of two discrete stochastic variables as a measure of their distance (Văduva and Resteanu, 2009; Nowak *et al*, 2004). However, the difference between the expectations of two different discrete stochastic variables may be 0 when this measure is used. Liu (1997) proposed a new distance measure as a remedy of the measure based on expectations. However, the distance between a discrete stochastic variable and itself may not be 0 with this measure. Cha (2007) presented a survey of different distance measures between two discrete stochastic variables proposed in the literature. These distance measures only consider the probability distributions but ignore the values of the stochastic variables. The new metric measuring the distance between two discrete stochastic variables proposed in this study defined in the following overcomes these limitations.

Definition 5. Let ξ_1 and ξ_2 be any two discrete stochastic variables with probability distributions $P(\xi_1)$ and $P(\xi_2)$. The distance between ξ_1 and ξ_2, denoted by $d(\xi_1,\xi_2)$, is defined in (11) in the following

$$d(\xi_1,\xi_2)=\sqrt{\frac{1}{2}[P(\xi_1)-P(\xi_2)]B[P(\xi_1)-P(\xi_2)]^T} \tag{11}$$

where

$$P(\xi_1)-P(\xi_2)=(P_1(\xi_1)-P_1(\xi_2),P_2(\xi_1)-P_2(\xi_2),\dots,P_l(\xi_1)-P_l(\xi_2)) \tag{12}$$

and each element b_{ij} of the matrix $B=(b_{ij})_{l\times l}$ is defined as

$$b_{ij}=x_l^{\,2}-(x_i-x_j)^2\,,\quad i,\ j=1,2,\dots,l\,. \tag{13}$$

Obviously, B is a symmetric positive definite matrix, *i.e.*, $\frac{1}{2}[P(\xi_1)-P(\xi_2)]B[P(\xi_1)-P(\xi_2)]^T\geq 0$. It can be seen that the smaller the difference between x_i and x_j is, the larger the value of b_{ij} is. Each b_{ij} has the following properties:

(1) $0\leq b_{ij}\leq x_l^{\,2}$;

(2) $b_{ij}=b_{ji}$;

(3) $b_{ii}=x_l^{\,2}=\max\{b_{ij}\mid i,\ j=1,2,\dots,l\}$;

(4) $b_{i'j'}>b_{i''j''}$ if $j''<j'<i'$, if $j''>j'>i'$, if $i''<i'<j'$, or if $i''>i'>j'$, *i.e.*, the further away an element is from the main diagonal, the smaller its values is.

The following example illustrates the calculation of the distance between two stochastic variables.

Example 2. For the two stochastic variables ξ_1 and ξ_2, their probability distributions are

$$P(\xi_1)=\begin{cases}0, & \xi_1=1\\ 0.5, & \xi_1=2\\ 0.3, & \xi_1=3\\ 0, & \xi_1=4\\ 0.2, & \xi_1=5\end{cases}\ \text{and}\ P(\xi_2)=\begin{cases}0, & \xi_2=1\\ 0, & \xi_2=2\\ 0.5, & \xi_2=3\\ 0.5, & \xi_2=4\\ 0, & \xi_2=5\end{cases}.$$

According to (12), $P(\xi_1)-P(\xi_2)=(0,0.5,-0.2,-0.5,0.2)$ and according to (13), the matrix B is

$$B = \begin{bmatrix} 25 & 24 & 21 & 16 & 9 \\ 24 & 25 & 24 & 21 & 16 \\ 21 & 24 & 25 & 24 & 21 \\ 16 & 21 & 24 & 25 & 24 \\ 9 & 16 & 21 & 24 & 25 \end{bmatrix}.$$

According to (11), the distance between ξ_1 and ξ_2 is then $d(\xi_1, \xi_2) = 0.6$.

The distance measure $d(\xi_1, \xi_2)$ between ξ_1 and ξ_2 defined in (11) has all the properties of a metric, *i.e.*, non-negativity, symmetry and triangle inequality. Furthermore, when ξ_1 and ξ_2 are deterministic, $d(\xi_1, \xi_2)$ reduces to the Euclidian distance. These properties are stated formally in the following.

Property 1 (Non-negativity). For any two discrete stochastic variables ξ_1 and ξ_2, $d(\xi_1, \xi_2) \geq 0$ and $d(\xi_1, \xi_2) = 0$ if and only if $P(\xi_1) = P(\xi_2)$.

Property 2 (Symmetry). For any two discrete stochastic variables ξ_1 and ξ_2, $d(\xi_1, \xi_2) = d(\xi_2, \xi_1)$.

Property 3 (Triangle inequality). For any three discrete stochastic variables ξ_1, ξ_2 and ξ_3, $d(\xi_1, \xi_2) + d(\xi_2, \xi_3) \geq d(\xi_1, \xi_3)$.

Property 4. If both ξ_1 and ξ_2 are deterministic, then the distance $d(\xi_1, \xi_2)$ is identical to the Euclidean distance.

The proofs of Properties 3 and 4 are given in the Appendix.

5. The method for solving the stochastic MADM problem

Consider a stochastic MADM problem. Let $\boldsymbol{A} = \{A_1, A_2, \cdots, A_m\}$ with $m \geq 2$ be a finite set of alternatives, and $\boldsymbol{Q} = \{Q_1, Q_2, \cdots, Q_n\}$ with $n \geq 2$ be a finite set of attributes. Let $\boldsymbol{w} = (w_1,\ w_2,\ \cdots,\ w_n)^T$, with $0 \leq w_j \leq 1$ for all $j = 1, \cdots, n$ and $\sum_{j=1}^{n} w_j = 1$, be an attribute weighting vector, where w_j is the weight assigned to attribute Q_j. Also let $Z = (z_{ij})_{m \times n}$ be the decision matrix, where z_{ij} is the evaluation of alternative A_i on attribute Q_j. Each z_{ij} is a discrete stochastic variable with a domain X_j^+ such that $X_j^+ = \{x_{j1}, x_{j2}, \ldots, x_{jl_j}\}$ with $x_{j1} < x_{j2} < \cdots < x_{jl_j}$ and $x_{j1}, x_{j2}, \ldots, x_{jl_j} > 0$, where l_j is the number of possible values of attribute Q_j. For a real life problem, the values of the elements of X_j^+ are the possible scores assigned to the alternatives on attribute Q_j. The domain X_j^+ can be different for different attributes, *i.e.*, there exist $Q_{j'}, Q_{j''} \in \boldsymbol{Q}$, with $j' \neq j''$ such that $X_{j'}^+ \neq X_{j''}^+$ and $l_{j'} \neq l_{j''}$. The probability that z_{ij} is equal to x_{jv} is represented by $P_v(z_{ij})$, with $P_v(z_{ij}) \geq 0$ for all $v = 1, 2, \ldots, l_j$ and $\sum_{v=1}^{l_j} P_v(z_{ij}) = 1$. The probability distributions of the evaluations of the m alternatives on each attribute Q_j are usually represented in the form as shown in Table 2. Each column of the table represents the probability distribution $P_v(z_{ij})$ of alternative A_i over the domain X_j^+. In a real life problem, these probabilities are obtained by the corresponding relative frequencies.

Table 2. Evaluations of the m alternatives on attribute Q_j

X_j^+	A_1	A_2	$\cdots$	A_m
x_{j1}	$P_1(z_{1j})$	$P_1(z_{2j})$	$\cdots$	$P_1(z_{mj})$
x_{j2}	$P_2(z_{1j})$	$P_2(z_{2j})$	$\cdots$	$P_2(z_{mj})$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
x_{jl_j}	$P_{l_j}(z_{1j})$	$P_{l_j}(z_{2j})$	$\cdots$	$P_{l_j}(z_{mj})$

The stochastic MADM problem concerned in this study is to rank the alternatives in the finite set $\boldsymbol{A}$. In the proposed method for the stochastic MADM problem, the domains of various attributes are normalized if they are different, the ideal and nadir solutions are identified, the distances of each alternative from the ideal and nadir solutions are calculated, the relative closeness of each alternative is determined, and the alternatives are finally ranked. The method based on the ideal and nadir solutions for the stochastic MADM problem is described in more detail in the following.

5.1 Normalization of attribute values

Different attributes may often be on different scales, *i.e.*, having different domains, for discrete stochastic MADM problems. To handle the attribute values with different scales in such cases, the attribute values need to be normalized into the same scale, *e.g.*, from 0 to 1. Let x'_{jv} denote the normalized value of x_{jv} in the domain X_j^+ for attribute j. Then x'_{jv} is given by

$$x'_{jv} = \frac{x_{jv}}{x_{jl_j}}, \quad v = 1, 2, \ldots, l_j \text{ and } j = 1, 2, \ldots, n. \tag{14}$$

Each element in the domain $X_j^+ = \{x_{j1}, x_{j2}, \ldots, x_{jl_j}\}$ is converted to an element in the domain $X_j'^+ = \{x'_{j1}, x'_{j2}, \ldots, x'_{jl_j}\}$ according to (14) in the normalization process. The normalization of the attribute values does not change the probabilities of the attribute values. Let z'_{ij} be the normalized attribute value of alternative A_i on attribute Q_j and $P'_v(z'_{ij})$ be the probability that z'_{ij} is equal to x'_{jv}. The probabilities then satisfy $P'_v(z'_{ij}) = P_v(z_{ij})$ and $\sum_{v=1}^{l_j} P'_v(z'_{ij}) = 1$.

The normalization only ensures the commensurability among the domains of different attributes, but the different attributes still have their values in their individual domains. For the normalized attribute values of different attributes to be comparable, the union of all the normalized domains, donated by X^+, is used as the common domain, *i.e.*, $X^+ = X_1'^+ \cup X_2'^+ \cup \ldots \cup X_n'^+$.

Without loss of generality, assume the common domain has l elements, *i.e.*, $X^+ = \{x_1, x_2, \ldots, x_l\}$, and the normalized attribute z'_{ij} is treated as a discrete stochastic variable within X^+. Let $P_g(z'_{ij})$ be the probability that z'_{ij} is equal to x_g, for $g = 1, 2, \ldots, l$. Then $P_g(z'_{ij})$ is given by

$$P_g(z'_{ij}) = \begin{cases} P'_v(z'_{ij}), & \text{if } x_g \in X'^+_j \text{ and } x_g = x'_{jv} \\ 0, & \text{if } x_g \notin X'^+_j \end{cases}, \quad g = 1, 2, \ldots, l \text{ and } v \in \{1, 2, \ldots, l_j\}. \tag{15}$$

5.2 Identification of the ideal and nadir solutions

Because the ideal and nadir solutions are used as reference points in ranking the alternatives, appropriately defining them and identifying them are important steps in the proposed method. Let $Z^+ = (z_1^+, z_2^+, \ldots, z_n^+)^T$ and $Z^- = (z_1^-, z_2^-, \ldots, z_n^-)^T$ be the ideal and nadir solutions, where z_j^+ and z_j^- are the ideal and nadir variates of attribute Q_j, respectively. Both z_j^+ and z_j^- are discrete stochastic variables, and each of them has a set of values with an associated probability distribution. Let $P_v(z_j^+)$ and $P_v(z_j^-)$ denote the probabilities that z_j^+ and z_j^- are equal to x_v, where $P_v(z_j^+)$ and $P_v(z_j^-)$ are determined based on Definitions 2 and 3, respectively.

After the probability distributions of z_j^+ and z_j^- are determined for each attribute Q_j, for $j = 1, \cdots, n$, the ideal solution Z^+ and nadir solution Z^- are identified.

5.3 Distances from Z^+ and Z^- and the relative closeness of each alternative

Because z'_{ij}, z_j^+ and z_j^- are all discrete stochastic variables, the distances d_{ij}^+ and d_{ij}^- of z'_{ij} from z_j^+ and can be computed using (16) and (17) z_j^-, respectively, in the following that are directly from (11),

$$d_{ij}^+ = d(z'_{ij}, z_j^+) = \sqrt{\frac{1}{2}[P(z'_{ij}) - P(z_j^+)]B[P(z'_{ij}) - P(z_j^+)]^T}, \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \cdots, n \tag{16}$$

$$d_{ij}^- = d(z'_{ij}, z_j^-) = \sqrt{\frac{1}{2}[P(z'_{ij}) - P(z_j^-)]B[P(z'_{ij}) - P(z_j^-)]^T}, \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \cdots, n, \tag{17}$$

where $P(z'_{ij}) - P(z_j^+) = (P_1(z'_{ij}) - P_1(z_j^+), P_2(z'_{ij}) - P_2(z_j^+), \ldots, P_l(z'_{ij}) - P_l(z_j^+))$ in (16) and $P(z'_{ij}) - P(z_j^-) =$ $(P'_1(z_{ij}) - P_1(z_j^-),\ P_2(z'_{ij}) - P_2(z_j^-), \ldots,\ P_l(z'_{ij}) - P_l(z_j^-))$ in (17). The components b_{ij} of B are defined in (13).

The distances d_i^+ and d_i^- of each alternative A_i from Z^+ and Z^- are the weighted sum of the individual distances d_{ij}^+ and d_{ij}^-. They are calculated with (18) and (19), respectively, in the following

$$d_i^+ = \sqrt{\sum\nolimits_{j=1}^{n} w_j{}^2 d_{ij}^{+2}}, \quad i = 1, 2, \ldots, m \tag{18}$$

$$d_i^- = \sqrt{\sum\nolimits_{j=1}^{n} w_j{}^2 d_{ij}^{-2}}, \quad i = 1, 2, \ldots, m. \tag{19}$$

With the results in (18) and (19), the relative closeness c_i^* of each alternative A_i is calculated in (20) in the following

$$c_i^* = \frac{d_i^-}{d_i^- + d_i^+}, \quad i = 1, 2, \ldots, m. \tag{20}$$

5.4 Rankings of the alternatives

The larger the relative closeness c_i^* is, the better the alternative A_i is. Therefore, the alternatives are ranked using their relative closeness c_i^* as defined in (20).

The method for the stochastic MADM problem based on the ideal and nadir solutions is given step by step in the following.

Step 1. Normalize the attribute value z_{ij} into z'_{ij} using (14) and (15) if different attributes have different domains.

Step 2. Determine the probability distributions of the components of the ideal solution Z^+ and the nadir solution Z^- using definitions 2 and 3, respectively.

Step 3. Calculate the distances of each alternative from the ideal solution d_i^+ and from the nadir solution d_i^- using (18) and (19), respectively.

Step 4. Calculate the relative closeness c_i^* for each alternative A_i for $i = 1, \cdots, m$ using (20).

Step 5. Rank the alternatives in descending order of c_i^* for $i = 1, \cdots, m$.

6. Illustrative examples

Two illustrative examples are presented in this section. The main purpose of these examples is to show the effectiveness and to demonstrate the use of the proposed method. The results obtained with the proposed method are also compared with those of some existing methods.

6.1 An example with the same attribute domain

In this example, the problem is to select the most desirable computer development project(s). Because the attributes in this example have the same domain, normalization of the attribute values is not necessary. The problem was previously studied by Martel and Zaras (1995) and Nowak (2004). The result obtained with the method proposed in this study is compared with that obtained with the ELECTRE-III method (Roy, 1985, 1991; Nowak, 2004).

In the problem, $m = 10$ computer development projects with $n = 4$ attributes are considered. The computer development projects are represented by the alternatives A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9 and A_{10}. The attributes are Q_1: personal resources effort; Q_2: discounted profit; Q_3: chances of success, and Q_4: technological orientation. The weights assigned to the attributes are $w_1 = 0.09$, $w_2 = 0.55$, $w_3 = 0.27$ and $w_4 = 0.09$.

Each attribute is evaluated on a scale from 1 to 10, *i.e.*, a score, with 1 being the lowest and 10 the highest, *i.e.*, $X^+ = \{1, 2, \cdots, 10\}$ with $l = 10$, by 7 experts. The evaluations of the alternatives on the attributes

Table 3. Evaluations of the alternatives on attribute Q_1

Score (X^+)	Alternative									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1							1/7			
2	3/7	1/7						1/7		1/7
3	1/7				1/7			2/7		2/7
4		2/7						2/7		2/7
5	2/7	1/7	3/7	1/7			3/7	1/7	2/7	1/7
6		2/7	1/7		2/7	1/7	1/7	1/7	1/7	
7	1/7		1/7	1/7	2/7	2/7			3/7	1/7
8		1/7	2/7	1/7		2/7	1/7		1/7	
9				3/7	2/7					
10				1/7		2/7	1/7			

Table 4. Evaluations of the alternatives on attribute Q_2

Score (X^+)	Alternative									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1							1/7	3/7		
2	2/7						3/7	3/7		1/7
3	1/7			1/7		4/7	1/7		1/7	
4				1/7				1/7	1/7	
5	2/7				1/7		1/7			
6		1/7	1/7	1/7	2/7		1/7		1/7	
7		1/7			1/7	1/7			4/7	2/7
8	1/7	3/7	2/7	3/7	2/7	2/7				3/7
9	1/7	2/7	3/7	1/7	1/7					
10			1/7							1/7

Table 5. Evaluations of the alternatives on attribute Q_3

Score (X^+)	Alternative									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1								2/7		1/7
2							3/7	1/7		2/7
3	1/7			1/7			1/7	4/7	1/7	
4	3/7					1/7	1/7		2/7	
5		1/7				1/7	2/7		2/7	2/7
6	1/7									2/7
7		1/7		1/7	2/7				2/7	
8	1/7	2/7	4/7	2/7	3/7	2/7				
9	1/7	3/7	3/7	1/7	1/7	1/7				
10				2/7	1/7	2/7				

are presented in Tables 3-6. Each column of a table lists the relative frequencies, representing corresponding probabilities, that each alternative A_i is assigned the different values in X^+ for one of the $n = 4$ attributes.

Because all $n = 4$ attributes share the same domain in this example, the normalization step, *i.e.*, Step 1 in the proposed method, is not required. According to definitions 2 and 3, the probability distributions of the components of the ideal and nadir solutions are determined as shown in Tables 7 and 8, respectively.

Table 6. Evaluations of the alternatives on attribute Q_4

Score (X^+)	Alternative									
	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1								2/7		
2									1/7	
3	3/7						1/7			
4								1/7	1/7	
5	2/7					1/7	1/7	2/7		
6			1/7		1/7	1/7		1/7	3/7	3/7
7			1/7		1/7	1/7				1/7
8	1/7	4/7	3/7	3/7	3/7	2/7	3/7	1/7	1/7	1/7
9		2/7		1/7	1/7	1/7	1/7			1/7
10	1/7	1/7	2/7	3/7	1/7	1/7	1/7		1/7	1/7

Table 7. The probability distributions of the components of the ideal solution Z^+

Ideal solution	Score									
	1	2	3	4	5	6	7	8	9	10
z_1^+								2/7	3/7	2/7
z_2^+								3/7	3/7	1/7
z_3^+								2/7	3/7	2/7
z_4^+								2/7	2/7	3/7

Table 8. The probability distributions of the components of the nadir solution Z^-

Nadir solution	Score									
	1	2	3	4	5	6	7	8	9	10
z_1^-	1/7	3/7	2/7	1/7						
z_2^-	3/7	3/7	1/7							
z_3^-	2/7	3/7	2/7							
z_4^-	2/7	1/7	3/7	1/7						

Table 9. Distances of the $m = 10$ alternatives from the ideal and nadir solutions

Alternative	d_i^+	d_i^-
A_1	2.3896	1.9810
A_2	0.6706	3.7782
A_3	0.3409	4.1273
A_4	1.2208	3.2018
A_5	0.9977	3.4062
A_6	1.5789	2.9722
A_7	3.5175	0.9482
A_8	4.2326	0.2378
A_9	1.9415	2.4659
A_{10}	1.7033	3.0641

From (18) and (19), the distances of each alternative from the ideal and the nadir solutions are calculated as shown in Table 9.

According to (20), the relative closeness of each alternative is calculated as follows

$c_1^* = 0.4532$; $c_2^* = 0.8492$; $c_3^* = 0.9237$; $c_4^* = 0.7239$; $c_5^* = 0.7735$;

$c_6^* = 0.6531$; $c_7^* = 0.2123$; $c_8^* = 0.0532$; $c_9^* = 0.5595$; $c_{10}^* = 0.6427$.

Therefore, the rankings of the alternatives are $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_6 \succ A_{10} \succ A_9 \succ A_1 \succ A_7 \succ A_8$.

The result of the method proposed in this study is compared with that of the extended ELECTRE-III method (Nowak, 2004) in Table 10. In the ELECTRE-III method as presented in Nowak (2004), the preference and veto thresholds of these four attributes are $p_j = 1$ and $v_j = 3$, for $j = 1,2,3,4$. The SD relations are verified first. The deviations between the mathematical expectations of stochastic evaluations are then calculated. The values of concordance and discordance indices are finally obtained by comparing the deviations with the preference and veto thresholds.

Table 10. Comparison of results of the two methods

Methods	Ranking of alternatives
The proposed method	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_6 \succ A_{10} \succ A_9 \succ A_1 \succ A_7 \succ A_8$
The ELECTRE-III method	$A_3 \succ A_2 \succ A_5 \sim A_4 \succ A_6 \succ A_{10} \sim A_9 \succ A_1 \succ A_7 \succ A_8$

Compared with the ELECTRE-III method, the method developed in this study has three good characteristics:

Clear rankings of the alternatives. As shown in Table 10, the results of the two methods are pretty consistent. Each alternative has a distinct ranking with the method proposed in this study. However, the rankings of alternatives A_4 and A_5 and those of alternatives A_9 and A_{10} cannot be distinguished with the ELECTRE-III method. The main reason is that the probability distributions of the attribute values need to be transformed into mathematical expectations in the ELECTRE-III method. This transformation, however, causes information loss. Moreover, if SD relations cannot be verified, the values of both the concordance and discordance indices are 0, implying that these indices are meaningless.

Independent of the preference and veto thresholds. The method proposed in this study does not need the preference and veto thresholds. However, the rankings of the alternatives obtained are very close to those using the preference and veto thresholds in the ELECTRE-III method. Subjective judgments are introduced into the decision process when the preference and veto thresholds are used. Actually, using different preference and veto thresholds may lead to different rankings of the alternatives. Hence, the DM must specify accurate and reasonable preference and veto thresholds in the ELECTRE-III method. Unfortunately, these thresholds are often very difficult to specify.

Independent of the SD relations. Furthermore, the ELECTRE-III method for solving the stochastic MADM problem needs to verify SD relations but the method proposed in this study does not. As a result, the proposed method in this study simplifies the computation process.

6.2 An example with different attribute domains

The main purpose of this example is to show how the evaluations on the attributes with different domains are normalized and the union of the different domains is used as the common domain. Suppose an overseas investor is interested in selecting a company with the most potential to invest in SY city in China. A total of $m = 6$ companies, as alternatives, with $n = 3$ attributes are considered. The $m = 6$ companies are A_1 : an auto manufacturer; A_2 : a pharmaceutical firm; A_3 : a food processing company; A_4 : a logistics company; A_5 : an apparel manufacturer, and A_6 : a computer software company. The attributes are Q_1 : return on investment; Q_2 : growth, and Q_3 : impact on environment. The $n = 3$ attributes are all assumed to be of benefit type. The weights assigned to the attributes are $w_1 = 0.3$, $w_2 = 0.5$ and $w_3 = 0.2$. Because of the difference in the characteristics of the attributes, the scales on different attributes are different. For attribute Q_1, the domain is $X_1^+ = \{1, 2, 3, 4, 5\}$ with 1 being the worst and 5 being the best, while for attributes Q_2 and Q_3, the domains are $X_2^+ = X_3^+ = \{1, 2, 3, 4, 5, 6, 7\}$ with 1 being the worst and 7 being the best. Because the attributes in this example have different domains, the normalization step is needed to put the evaluations under the same scale. This overseas investor invited 5 experts to provide their evaluations of the alternatives for each attribute. The evaluations of the alternatives on the attributes are presented in Tables 11-13.

Table 11. Evaluations of the alternatives on attribute Q_1

Score (X_1^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1	1/5	1/5	2/5	1/5		
2	1/5	1/5	3/5			
3	1/5	1/5		3/5		2/5
4	2/5	2/5		1/5	2/5	0
5					3/5	3/5

Table 12. Evaluations of the alternatives on attribute Q_2

Score (X_2^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1	1/5	2/5	1/5		3/5	
2						
3	1/5					1/5
4	1/5		2/5		1/5	1/5
5	2/5	1/5	2/5	1/5	1/5	2/5
6		1/5		2/5		1/5
7		1/5		2/5		

The attribute values are normalized using (14) and (15). The union of the domains is obtained and is used as the common domain X^+. The normalized attribute values in the common domain X^+ are shown in Tables 14-16.

Table 13. Evaluations of the alternatives on attribute Q_3

Score (X_3^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1				3/5	2/5	
2	1/5	1/5			2/5	
3				2/5	1/5	1/5
4	2/5	2/5	1/5			1/5
5			1/5			2/5
6	2/5	2/5				1/5
7			3/5			

Table 14. Normalized evaluations of the alternatives on attribute Q_1

Score (X^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1/7						
1/5	1/5	1/5	2/5	1/5		
2/7						
2/5	1/5	1/5	3/5			
3/7						
4/7						
3/5	1/5	1/5		3/5		2/5
5/7						
4/5	2/5	2/5		1/5	2/5	
6/7						
1					3/5	3/5

Table 15. Normalized evaluations of the alternatives on attribute Q_2

Score (X^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1/7	1/5	2/5	1/5		3/5	
1/5						
2/7						
2/5						
3/7	1/5					1/5
4/7	1/5		2/5		1/5	1/5
3/5						
5/7	2/5	1/5	2/5	1/5	1/5	2/5
4/5						
6/7		1/5		2/5		1/5
1		1/5		2/5		

According to definitions 2 and 3, the probability distributions of the components of the ideal and nadir solutions are determined as shown in Tables 17 and 18, respectively.

From (18) and (19), the distances of each alternative from the ideal and the nadir solutions are calculated as shown in Table 19.

Table 16. Normalized evaluations of the alternatives on attribute Q_3

Score (X^+)	Alternatives					
	A_1	A_2	A_3	A_4	A_5	A_6
1/7				3/5	2/5	
1/5						
2/7	1/5	1/5			2/5	
2/5						
3/7				2/5	1/5	1/5
4/7	2/5	2/5	1/5			1/5
3/5						
5/7			1/5			2/5
4/5						
6/7	2/5	2/5				1/5
1			3/5			

Table 17. The probability distributions of the components of the ideal solution Z^+

Ideal solution	Score										
	1/7	1/5	2/7	2/5	3/7	4/7	3/5	5/7	4/5	6/7	1
z_1^+									2/5		3/5
z_2^+								1/5		2/5	2/5
z_3^+										2/5	3/5

Table 18. The probability distributions of the components of the nadir solution Z^-

Nadir solution	Score										
	1/7	1/5	2/7	2/5	3/7	4/7	3/5	5/7	4/5	6/7	1
z_1^-		2/5		3/5							
z_2^-	2/5			1/5	2/5						
z_3^-	3/5		2/5								

Table 19. Distances of the $m = 6$ alternatives from the ideal and nadir solutions

Alternative	d_i^+	d_i^-
A_1	0.2238	0.1520
A_2	0.2208	0.1726
A_3	0.2492	0.1761
A_4	0.1746	0.2976
A_5	0.3041	0.1812
A_6	0.1300	0.2511

According to (20), the relative closeness of each alternative is calculated as follows

$c_1^* = 0.4045$; $c_2^* = 0.4623$; $c_3^* = 0.4140$; $c_4^* = 0.6303$; $c_5^* = 0.3733$; $c_6^* = 0.6589$.

Therefore, the rankings of the alternatives are $A_6 \succ A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$.

The result obtained with the method proposed in this study is compared with that of the method based on SDD (Zhang *et al*, 2010). The rankings of the alternatives by the method based on SDD are $A_6 \succ A_4 \succ A_3$

$\succ A_1 \succ A_2 \succ A_5$. The rankings of alternatives A_1 and A_2 are reversed in the method based on SDD from those by the method proposed in this study. The reason may be explained through the following example. Look at z'_{12} and z'_{22} only in Table 15. Since $F_{z'_{12}}(1/7) = 1/5 \le F_{z'_{22}}(1/7) = 2/5$ and $F_{z'_{12}}(4/7) = 3/5 \ge F_{z'_{22}}(4/7) = 2/5$, $\int_0^{1/7} F_{z'_{12}}(x)dx = 1/70 \le \int_0^{1/7} F_{z'_{22}}(x)dx = 1/35$ and $\int_0^{4/7} F_{z'_{12}}(x)dx = 6/35 \ge \int_0^{4/7} F_{z'_{22}}(x)dx = 4/35$, $\int_0^{1/7}\int_0^{x} F_{z'_{12}}(t)dtdx = 1/735 \le \int_0^{1/7}\int_0^{x} F_{z'_{22}}(t)dtdx = 2/735$ and $\int_0^{4/7}\int_0^{x} F_{z'_{12}}(t)dtdx = 16/245 \ge \int_0^{4/7}\int_0^{x} F_{z'_{22}}(t)dtdx = 32/685$, the SD relation for $F_{z'_{12}}(x)$ and $F_{z'_{22}}(x)$ cannot be determined by the method based on SDD. According to the method in Zhang *et al* (2010), the SDD for z'_{12} over z'_{22} is $\psi(F_{z'_{12}}(x) SD F_{z'_{22}}(x)) = \psi(F_{z'_{12}}(x) SD F_{z'_{22}}(x)) = 0$, although z'_{12} and z'_{22} are two different stochastic variables. With the method proposed in this study, the two distances from the ideal solution are $d(z'_{12}, z_2^+) = 0.3714$ and $d(z'_{22}, z_2^+) = 0.3143$ and the two distances from the nadir solution are $d(z'_{12}, z_2^-) = 0.2057$ and $d(z'_{22}, z_2^-) = 0.2629$, respectively. The relative closeness of each alternative based only on attribute Q_2 are $c_1^* = 0.3564$ and $c_2^* = 0.4555$. Obviously, $z'_{22} \succ z'_{12}$ is determined. Therefore, comparing with the method based on SDD (Zhang *et al*, 2010), the comparison result of attribute values using the method proposed in this study is clearer and subtler. Furthermore, the method proposed in this study does not require the verification of the SD relations and, therefore, simplifies the computation process.

7. Conclusions

This paper presents a novel method for solving the stochastic MADM problem. This method simultaneously considers distances of the alternatives both from the ideal and nadir solutions. The relative closeness of an alternative is obtained by combining the distances from the ideal and nadir solutions. The alternatives are ranked using their relative closeness. The probability distributions of ideal and nadir variates, as components of the ideal and nadir solutions, are defined and their properties and rationales are discussed. A metric measuring the distance between two discrete stochastic variables is defined. Some properties of the metric are discussed. The probability distributions of the ideal and nadir solutions in stochastic MADM problems are defined.

Compared with existing methods for solving the stochastic MADM problem, the proposed method has distinct characteristics. It overcomes some limitations in the existing methods, *i.e.*, it does not need to construct a multi-attribute utility function, does not need to transform the stochastic information into other types of information and does not need to identify the SD relations. Furthermore, this new method for the stochastic MADM problem based on the ideal and nadir solutions has a clear logic, a simple computation process and a broad applicability. Since the proposed method can obtain clear alternative rankings, it gives the DMs one more tool for solving stochastic MADM problems.

For future research, the proposed method can be extended to more complex stochastic MADM problems. One type of such problems is when the attribute values are in the form of continuous stochastic variables.

Another type of such problems is when the attribute values are in the form of discrete stochastic variables with uncertain probability information.

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Appendix

Proof of Proposition 1. In the following, the property that $F_+(x)$ FSD $F_u(x)$ is proved. The other properties can be proved analogously.

The cumulative probability distribution functions of ξ_u and ξ^+, $F_u(x)$ and $F_+(x)$, are given by

$$F_u(x)=\begin{cases}0, & x<x_1\\ P_1(\xi_u), & x_1\le x<x_2\\ \sum_{v=1}^{2}P_v(\xi_u), & x_2\le x<x_3\\ \vdots & \vdots\\ \sum_{v=1}^{l-1}P_v(\xi_u), & x_{l-1}\le x<x_l\\ 1, & x\ge x_l\end{cases} \text{ and } F_+(x)=\begin{cases}0, & x<x_1\\ P_1(\xi^+), & x_1\le x<x_2\\ \sum_{v=1}^{2}P_v(\xi^+), & x_2\le x<x_3\\ \vdots & \vdots\\ \sum_{v=1}^{l-1}P_v(\xi^+), & x_{l-1}\le x<x_l\\ 1, & x\ge x_l\end{cases}.$$

There are two cases to be considered.

Case 1: $P_v(\xi^+)=\max_u\{P_v(\xi_u)\,|\,u=1,2,\ldots,h\}$ for $v=2,\ldots,l$, $P_1(\xi^+)=1-\sum_{v=2}^{l}P_v(\xi^+)$ and $P_1(\xi^+)>0$.

Obviously, $\sum_{v=1}^{l}P_v(\xi_u)=\sum_{v=1}^{l}P_v(\xi^+)=1$, for $u=1,2,\ldots,h$.

For $u=1,2,\ldots,h$ and $g=1,\ldots,l-1$, it can be seen that $\sum_{v=g+1}^{l}P_v(\xi_u)\le\sum_{v=g+1}^{l}P_v(\xi^+)$. Then, $\sum_{v=1}^{l}P_v(\xi_u)-\sum_{v=g+1}^{l}P_v(\xi_u)\ge\sum_{v=1}^{l}P_v(\xi^+)-\sum_{v=g+1}^{l}P_v(\xi^+)$, *i.e.*, $\sum_{v=1}^{g}P_v(\xi_u)\ge\sum_{v=1}^{g}P_v(\xi^+)$. Furthermore, $F_u(x)=\sum_{v=1}^{g}P_v(\xi_u)\ge\sum_{v=1}^{g}P_v(\xi^+)=F_+(x)$ for $x_g\le x<x_{g+1}$.

Particularly, for $g=l$, *i.e.*, $x=x_g=x_l$, $F_u(x)=\sum_{v=1}^{g}P_v(\xi_u)=\sum_{v=1}^{g}P_v(\xi^+)=F_+(x)=1$. Therefore, $F_u(x)\ge F_+(x)$ for all $x_1\le x\le x_l$. According to Definition 1, $F_+(x)$ FSD $F_u(x)$.

Case 2: $P_v(\xi^+)=\max_u\{P_v(\xi_u)\,|\,u=1,2,\ldots,h\}$ for $v=k+2,\ldots,l$, $P_{k+1}(\xi^+)=1-\sum_{v=k+2}^{l}P_v(\xi^+)$, and $P_v(\xi^+)=0$ for $v=1,\ldots,k$ and $1\le k<l$. Obviously, $P_v(\xi^+)\ge 0$ for $v=k+1\ldots,l$, and $\sum_{v=k+1}^{l}P_v(\xi^+)=1$.

For ξ_u, a new discrete stochastic variable ξ_u' is introduced, where $P_v(\xi_u')=P_v(\xi_u)$ for $v=k+2,\ldots,l$, and $P_{k+1}(\xi_u')=1-\sum_{v=k+2}^{l}P_v(\xi_u')$. It can be seen that $P_{k+1}(\xi_u')=1-\sum_{v=k+2}^{l}P_v(\xi_u)=\sum_{v=1}^{k+1}P_v(\xi_u)$ and $\sum_{v=k+1}^{l}P_v(\xi_u')=1$.

Let $F_u{}'(x)$ denote the cumulative probability distribution function of $\xi_u{}'$. $F_u{}'(x)$ is given by

$$F_u{}'(x)=\begin{cases}0, & x_1\le x<x_{k+1}\\ P_{k+1}(\xi_u{}'), & x_{k+1}\le x<x_{k+2}\\ \sum_{v=k+1}^{k+2}P_v(\xi_u{}'), & x_{k+2}\le x<x_{k+3}\\ \vdots & \vdots\\ \sum_{v=k+1}^{l-1}P_v(\xi_u{}'), & x_{l-1}\le x<x_l\\ 1, & x\ge x_l\end{cases}.$$

For $u=1,2,\ldots,h$, since $\sum_{v=k+1}^{l}P_v(\xi_u{}')=1$, then $P_v(\xi_u{}')=0$ for $v=1,\ldots,k$. Because $P_v(\xi_u)\ge 0$ for $v=1,\ldots,k$, then $\sum_{v=1}^{g}P_v(\xi_u)\ge\sum_{v=1}^{g}P_v(\xi_u{}')$ for $g=1,\ldots,k$. Furthermore, because $P_v(\xi_u{}')=P_v(\xi_u)$ for $v=k+2,\ldots,l$, then $\sum_{v=1}^{g}P_v(\xi_u)=\sum_{v=1}^{k+1}P_v(\xi_u)+\sum_{v=k+2}^{g}P_v(\xi_u)=\sum_{v=k+1}^{g}P_v(\xi_u{}')=\sum_{v=1}^{g}P_v(\xi_u{}')$ for $g=k+2,\ldots,l-1$.

Also because $\sum_{v=1}^{k+1}P_v(\xi_u{}')=\sum_{v=1}^{k}P_v(\xi_u{}')+P_{k+1}(\xi_u{}')=\sum_{v=1}^{k+1}P_v(\xi_u)$, then $\sum_{v=1}^{g}P_v(\xi_u)=\sum_{v=1}^{g}P_v(\xi_u{}')$ for $g=k+1,\ldots,l-1$. Therefore, the following can be obtained

$F_u(x)=\sum_{v=1}^{g}P_v(\xi_u)\ge\sum_{v=1}^{g}P_v(\xi_u{}')=F_u{}'(x)$, for $x_g\le x<x_{g+1}$ and $g=1,\ldots,k$.

$F_u(x)=\sum_{v=1}^{g}P_v(\xi_u)=\sum_{v=1}^{g}P_v(\xi_u{}')=F_u{}'(x)$, for $x_g\le x\le x_{g+1}$ and $g=k+1,\ldots,l-1$.

It can be seen that $F_u(x)\ge F_u{}'(x)$ for any $x_1\le x\le x_l$. According to Definition 4, $F_u{}'(x)$ FSD $F_u(x)$.

On the other hand, because $P_v(\xi_u{}')=P_v(\xi^+)=0$ for $v=1,\ldots,k$, $\sum_{v=1}^{g}P_v(\xi_u{}')=\sum_{v=1}^{g}P_v(\xi^+)=0$ can be obtained for $g=1,\ldots,k$.

Since $P_v(\xi_u{}')=P_v(\xi_u)\le P_v(\xi^+)$ for $v=k+2,\ldots,l$, then $\sum_{v=g+1}^{l}P_v(\xi_u{}')\le\sum_{v=g+1}^{l}P_v(\xi^+)$ for $g=k+1,\ \ldots,l-1$. Furthermore, by $\sum_{v=k+1}^{l}P_v(\xi_u{}')=\sum_{v=k+1}^{l}P_v(\xi^+)=1$, $\sum_{v=k+1}^{g}P_v(\xi_u{}')=$ $\sum_{v=k+1}^{l}P_v(\xi_u{}')-\sum_{v=g+1}^{l}P_v(\xi_u{}')\ge\sum_{v=k+1}^{l}P_v(\xi^+)-\sum_{v=g+1}^{l}P_v(\xi^+)=\sum_{v=k+1}^{g}P_v(\xi^+)$, for $g=k+1,\ \ldots,l-1$.

Therefore, the following can be obtained

$F_u{}'(x)=\sum_{v=1}^{g}P_v(\xi_u{}')=\sum_{v=1}^{g}P_v(\xi^+)=F_+(x)$, for $x_g\le x<x_{g+1}$ and $g=1,\ldots,k$.

$F_u{}'(x)=\sum_{v=k+1}^{g}P_v(\xi_u{}')\ge\sum_{v=k+1}^{g}P_v(\xi^+)=F_+(x)$, for $x_g\le x<x_{g+1}$ and $g=k+1,\ \ldots,l-1$.

$F_u{}'(x)=\sum_{v=1}^{g}P_v(\xi_u{}')=\sum_{v=1}^{g}P_v(\xi^+)=F_+(x)=1$, for $x=x_l$ and $g=l$.

It can be seen that, $F_u{}'(x)\ge F_+(x)$ for any $x_1\le x\le x_l$. According to Definition 1, $F_+(x)$ FSD $F_u{}'(x)$, for $u=1,2,\ldots,h$.

Because $F_+(x)$ FSD $F_u(x)$, $F_+(x)$ SSD $F_u(x)$ and $F_+(x)$ TSD $F_u(x)$, for $u=1,2,\ldots,h$, from Remarks 1 and 2. Analogously, $F_u(x)$ FSD $F_-(x)$, $F_u(x)$ SSD $F_-(x)$, and $F_u(x)$ TSD $F_-(x)$ can be proved for $u=1,2,\ldots,h$. □

Proof of Proposition 2. There are two cases to be considered.

Case 1: $P_v(\xi^+)=\max_u\{P_v(\xi_u)\,|\,u=1,2,\ldots,h\}$ for $v=2,\ldots,l$, $P_1(\xi^+)=1-\sum_{v=2}^{l}P_v(\xi^+)$ and $P_1(\xi^+)>0$.

Obviously, $P_v(\xi_u)\le P_v(\xi^+)$ for $v=2,\ldots,l$ and $\sum_{v=1}^{l}P_v(\xi^+)=1$. It can be seen that $\sum_{v=2}^{l}P_v(\xi_u)\le\sum_{v=2}^{l}P_v(\xi^+)$. Furthermore, $1-\sum_{v=2}^{l}P_v(\xi^+)=P_1(\xi^+)\le 1-\sum_{v=2}^{l}P_v(\xi_u)=P_1(\xi_u)$. Thus, $E(\xi^+)-E(\xi_u)=\sum_{v=2}^{l}(x_v-x_1)(P_v(\xi^+)-P_v(\xi_u))$. Because $x_1<x_v$, then $E(\xi^+)-E(\xi_u)\ge 0$.

Case 2: $P_v(\xi^+)=\max_u\{P_v(\xi_u)\,|\,u=1,2,\ldots,h\}$ for $v=k+2,\ \ldots,l$, $P_{k+1}(\xi^+)=1-\sum_{v=k+2}^{l}P_v(\xi^+)$, and $P_v(\xi^+)=0$ for $v=1,\ldots,k$ and $1\le k<l$. Obviously, $P_v(\xi^+)\ge 0$ for $v=k+1\ldots,l$ and $\sum_{v=k+1}^{l}P_v(\xi^+)=1$.

In order to compare $E(\xi^+)$ and $E(\xi_u)$, a new discrete stochastic variable ξ_u' is introduced, where $P_v(\xi_u')=P_v(\xi_u)$, for $v=k+2,\ldots,l$, and $P_{k+1}(\xi_u')=1-\sum_{v=k+2}^{l}P_v(\xi_u')$. It can be seen that $P_{k+1}(\xi_u')=\sum_{v=1}^{k+1}P_v(\xi_u)$ and $\sum_{v=k+1}^{l}P_v(\xi_u')=1$. Then, the expectation of ξ_u' can be given as follows.

$$E(\xi_u')=\sum_{v=1}^{l}x_vP_v(\xi_u')=\sum_{v=k+2}^{l}x_vP_v(\xi_u)+x_{k+1}\sum_{v=1}^{k+1}P_v(\xi_u).$$

Thus, $E(\xi_u')-E(\xi_u)=\sum_{v=1}^{l}x_vP_v(\xi_u')-\sum_{v=1}^{l}x_vP_v(\xi_u)=\sum_{v=1}^{k+1}P_v(\xi_u)(x_{k+1}-x_v)$.

Because $x_v\le x_{k+1}$ and $P_v(\xi_u)\ge 0$ for $v=1,\ldots,k+1$, then $E(\xi_u')-E(\xi_u)\ge 0$.

On the other hand, since $P_v(\xi_u')=P_v(\xi_u)$ and $P_v(\xi_u)\le P_v(\xi^+)$ for $v=k+2,\ldots,l$, then $P_v(\xi_u')\le P_v(\xi^+)$, for $v=k+2,\ldots,l$, and $\sum_{v=k+2}^{l}P_v(\xi_u)\le\sum_{v=k+2}^{l}P_v(\xi^+)$. Therefore, $1-\sum_{v=k+2}^{l}P_v(\xi_u)=P_{k+1}(\xi_u')\ge P_{k+1}(\xi^+)$ $=1-\sum_{v=k+2}^{l}P_v(\xi^+)$. Thus, $E(\xi^+)-E(\xi_u')=\sum_{v=k+2}^{l}(x_v-x_{k+1})(P_v(\xi^+)-P_v(\xi_u'))$.

Because $x_{k+1}<x_v$ and $P_v(\xi_u')\le P_v(\xi^+)$ for $v=k+2,\ldots,l$, then $E(\xi^+)-E(\xi_u')\ge 0$. Hence, $E(\xi^+)-E(\xi_u)\ge 0$. □

Proof of Property 3. For simplicity, let $\alpha=P(\xi_1)-P(\xi_2)=(\alpha_1,\alpha_2,\ldots,\alpha_l)$ and $\beta=P(\xi_2)-P(\xi_3)=(\beta_1,$ $\beta_2,\ldots,\beta_l)$. Hence, $\alpha+\beta=P(\xi_1)-P(\xi_3)=(\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots,\alpha_l+\beta_l)$, where $\alpha_v=P_v(\xi_1)-P_v(\xi_2)$, $\beta_v=P_v(\xi_2)-P_v(\xi_3)$, for $v=1,2,\ldots,l$. Obviously, $\sum_{v=1}^{l}\alpha_v=0$ and $\sum_{v=1}^{l}\beta_v=0$.

Since $B=B^T$ and $\alpha B\beta^T=(\alpha B\beta^T)^T$, then $\alpha B\beta^T=\beta B\alpha^T$. Therefore,

$$\left(d(\xi_1,\xi_2)+d(\xi_2,\xi_3)\right)^2-\left(d(\xi_1,\xi_3)\right)^2$$

$$=\left(\sqrt{\frac{1}{2}\alpha B\alpha^T}+\sqrt{\frac{1}{2}\beta B\beta^T}\right)^2-\left(\sqrt{\frac{1}{2}(\alpha+\beta)B(\alpha+\beta)^T}\right)^2$$

$$=\sqrt{\alpha B\alpha^T\beta B\beta^T}-\frac{1}{2}\alpha B\beta^T-\frac{1}{2}\beta B\alpha^T$$

$$=\sqrt{\alpha B\alpha^T\beta B\beta^T}-\alpha B\beta^T .$$

For $\alpha B\beta^T$, there are two cases to be considered:

Case 1: When $\alpha B\beta^T \le 0$. Obviously, $\sqrt{\alpha B\alpha^T\beta B\beta^T}-\alpha B\beta^T \ge 0$.

Case 2: When $\alpha B\beta^T > 0$. Since $B(\alpha^T\beta-\beta^T\alpha)B$ is an antisymmetric matrix, then

$$\alpha B\alpha^T\beta B\beta^T-(\alpha B\beta^T)^2$$

$$=\alpha B(\alpha^T\beta-\beta^T\alpha)B\beta^T$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{q=1}^{l}\sum_{v=1}^{l}\alpha_i b_{ij}(\alpha_j\beta_v-\alpha_v\beta_j)b_{qv}\beta_q$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{q=1}^{l}\sum_{v>j}^{l}(b_{ij}b_{qv}-b_{iv}b_{qj})\alpha_i\beta_q(\alpha_j\beta_v-\beta_j\alpha_v)$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{q>i}^{l}\sum_{v>j}^{l}(b_{ij}b_{qv}-b_{iv}b_{qj})(\alpha_i\beta_q-\alpha_q\beta_i)(\alpha_j\beta_v-\beta_j\alpha_v)$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{q>i}^{l}\sum_{v>j}^{l}(b_{ii}b_{qq}-b_{iq}b_{qi})(\alpha_i\beta_q-\alpha_q\beta_i)^2$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{q>i}^{l}\sum_{v>j}^{l}(b_{ii}{}^2-b_{iq}{}^2)(\alpha_i\beta_q-\alpha_q\beta_i)^2 .$$

Because $b_{ii}=x_l{}^2 \ge b_{iq}=x_l{}^2-(x_i-x_q)^2$ for $q,i=1, 2,\ldots,l$, then $\alpha B\alpha^T\beta B\beta^T-(\alpha B\beta^T)^2\ge 0$. Therefore, $\sqrt{\alpha B\alpha^T\beta B\beta^T}-\alpha B\beta^T \ge 0$, *i.e.*, $d(\xi_1,\xi_2)+d(\xi_2,\xi_3)\ge d(\xi_1,\xi_3)$. □

Proof of Property 4. Suppose ξ_1 and ξ_2 are deterministic with values x_v and x_t, respectively. Then the probability distributions of ξ_1 and ξ_2 can be written as

$$P(\xi_1)=\begin{cases}1, & \xi_1=x_v,\ v\in\{1,2,\ldots,l\}\\ 0, & \xi_1=x_k,\ k=1,2,\ldots,l,\ k\ne v\end{cases} \text{ and } P(\xi_2)=\begin{cases}1, & \xi_2=x_t,\ t\in\{1,2,\ldots,l\}\\ 0, & \xi_2=x_y,\ y=1,2,\ldots,l,\ y\ne t\end{cases}.$$

Without loss of generality, assume $v<t$. Hence, $P(\xi_1)-P(\xi_2)=(\underbrace{0,\ldots,0}_{v-1},1,\underbrace{0,\ldots,0}_{t-v-1},-1,\underbrace{0,\ldots,0}_{l-t})$. From (11)-(13), the distance between ξ_1 and ξ_2 is

$$d(\xi_1,\xi_2)=\sqrt{\frac{1}{2}[P(\xi_1)-P(\xi_2)]B[P(\xi_1)-P(\xi_2)]^T}$$

$$=\sqrt{\frac{1}{2}(\underbrace{0,\ldots,0}_{v-1},1,\underbrace{0,\ldots,0}_{t-v-1},-1,\underbrace{0,\ldots,0}_{l-t})B(\underbrace{0,\ldots,0}_{v-1},1,\underbrace{0,\ldots,0}_{t-v-1},-1,\underbrace{0,\ldots,0}_{l-t})^T}$$

$$=\sqrt{\frac{1}{2}[(x_l{}^2-x_l{}^2+(x_t-x_v)^2]-[x_l{}^2-(x_v-x_t)^2-x_l{}^2]}$$

$$=\| x_t-x_v \| .$$ □

References

Awasthi, A., Chauhan, S. S., Omrani, H., & Panahi, A. (2011). A hybrid approach based on SERVQUAL and fuzzy TOPSIS for evaluating transportation service quality. *Computers & Industrial Engineering*, 61(3), 637–646.

Bawa, V. S. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2(1), 95–121.

Brans, J. P. and Vincke, P. (1985). A preference ranking organization method (The PROMETHEE method for multiple criteria decision-making). *Management Science*, 31(6), 641–656.

Cha, S. H. (2007). Comprehensive survey on distance/similarity measures between probability density functions. *International Journal of Mathematical Models and Methods in Applied Sciences*, 4(1), 300–307.

Chuu, S. J. (2009). Selecting the advanced manufacturing technology using fuzzy multiple attributes group decision making with multiple fuzzy information. *Computers & Industrial Engineering*, 57(3), 1033–1042.

D'Avignon, G. and Vincke, P. (1988). An outranking method under uncertainty. *European Journal of Operational Research*, 185(36), 311–321.

Dymova, L., Sevastjanov, P. and Tikhonenko, A. (2013). An approach to generalization of fuzzy TOPSIS method. *Information Sciences*, 238, 149–162.

Fan, Z. P., Liu, Y. and Feng, B. (2010). A method for stochastic multiple criteria decision making based on pairwise comparisons of alternatives with random evaluations. *European Journal of Operational Research*, 207(2), 906–915.

Fan, Z. P., Ma, J., Jiang, Y. P., Sun, Y. H. and Ma, L. (2006). A goal programming approach to group decision making based on multiplicative preference and fuzzy preference. *European Journal of Operational Research*, 174(1), 311–321.

Fan, Z. P., Zhang, X., Liu, Y. and Zhang, Y. (2013). A method for stochastic multiple attribute decision making based on concepts of ideal and anti-ideal points. *Applied Mathematics and Computation*, 219(24), 11438–11450.

Hadar, J. and Russell, W. R. (1969). Rules for ordering uncertain prospects. *American Economic Review* 59(1), 25–34.

Hwang, C.-L. and Yoon, K. L. (1981). Multiple Attribute Decision Making: Methods and Applications: A State-of-the-Art Survey. Springer-Verlag, New York.

Jiang, Y. P., Fan, Z. P. and Ma, J. (2008). A method for group decision making with multi-granularity linguistic assessment information. *Information Sciences*, 178(4), 311–321.

Keeney, R. and Raiffa, H. (1976). Decisions with Multiple Objectives: Preferences and Value Tradeoffs. John Wiley & Sons, New York.

Khalili-Damghani, K., Sadi-Nezhad, S. and Tavana, M. (2013). Solving Multi-Period Project Selection Problems with Fuzzy Goal Programming Based on TOPSIS and a Fuzzy Preference Relation.

Information Sciences, 252, 42–61.

Kolli, S. and Parsaei, H. R. (1992). Multi-criteria analysis in the evaluation of advanced manufacturing technology using PROMETHEE. *Computers & Industrial Engineering*, 23(1–4), 455–458.

Lahdelma, R. and Salminen, P. (2009). Prospect theory and stochastic multicriteria acceptability analysis (SMAA). *Omega*, 37(5), 961–971.

Lahdelma, R., Salminen, P. and Hokkanen J.(2002). Locating a waste treatment facility by using stochastic multicriteria acceptability analysis with ordinal criteria. *European Journal of Operational Research*, 142(2), 345–356.

Leshno, M., and Levy, H. (2002). Preferred by "all" and preferred by "most" decision makers: almost stochastic dominance. *Management Science*, 48(8), 1074–1085.

Liu B, Random programming and Fuzzy Programming, Tsinghua Press, Beijing 1997.

Martel, J. and D'Avignon, G. (1982). Projects ordering with multicriteria analysis. *European Journal of Operational Research*, 179(10), 56–69.

Martel, J., D'Avignon, G. and Couillard, G. (1986). A fuzzy outranking relation in multicriteria decision making. *European Journal of Operational Research*, 183(25), 258–271.

Martel, J. M., and Zaras, K. (1995). Stochastic dominance in multicriteria analysis under risk. *Theory and Decision*, 39(1), 31–49.

Nowak, M. (2004). Preference and veto thresholds in multicriteria analysis based on stochastic dominance. *European Journal of Operational Research*, 158(2), 339–350.

Nowak, M. (2007). Aspiration level approach in stochastic MCDM problems. *European Journal of Operational Research*, 177(3), 1626–1640.

Roy, B. (1985). Méthodologie d'aide à la decision multicritère. *Economica*, Paris.

Roy, B. (1991). The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31(1), 49–73.

Sarker, R. A. and Quaddus, M. A. (2002). Modelling a nationwide crop planning problem using a multi-criteria decision making tool. *Computers & Industrial Engineering*, 42(2–4), 541–553.

Shih, H. S, Shyur, H. J. and Lee, E. S. (2007). An extension of TOPSIS for group decision making. *Mathematical and Computer Modelling*, 45(7), 801–813.

Stewart, T. (2005). Dealing with uncertainties in MCDA. In: Figueira, J., Greco, S., Ehrgott, M. (Eds.), *Multiple Criteria Decision Analysis: State of the Art Surveys.* Springer, (Chapter 11).

Văduva, I. and Resteanu, C. (2009). On solving stochastic MADM problems. *Yugoslav Journal of Operations Research*, 19(1), 75–83.

Yue, Z. (2012). Extension of TOPSIS to determine weight of decision maker for group decision making problems with uncertain information. *Expert Systems with Applications*, 39(7), 6343–6350.

Zaras, K. (1999). Rough approximation of pairwise comparisons described by multiattribute stochastic dominance. *Journal of Multi-Criteria Decision Analysis*, 8(5), 291–297.

Zaras, K. (2001). Rough approximation of a preference relation by a multi-attribute stochastic dominance for determinist and stochastic evaluation problems. *European Journal of Operational Research*, 130(2), 305–

314.

Zaras, K. (2004). Rough approximation of a preference relation by a multi-attribute dominance for deterministic, stochastic and fuzzy decision problems. *European Journal of Operational Research*, 159(1), 196–206.

Zhang, Y., Fan, Z. P. and Liu, Y. (2010). A method based on stochastic dominance degrees for stochastic multiple criteria decision making. *Computers & Industrial Engineering*, 58(4), 544–552.

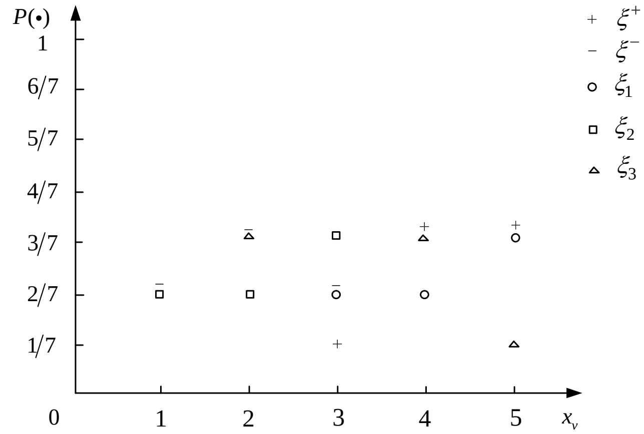

Figure 1. Probability distributions of ξ_1, ξ_2 and ξ_3 as well as of ξ^+ and ξ^-