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Robust Optimization for Interactive Multiobjective Programming with Imprecise Information Applied to R&D Project Portfolio Selection

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Abstract

A multiobjective binary integer programming model for R&D project portfolio selection with competing objectives is developed when problem coefficients in both objective functions and constraints are uncertain. Robust optimization is used in dealing with uncertainty while an interactive procedure is used in making tradeoffs among the multiple objectives. Robust nondominated solutions are generated by solving the linearized counterpart of the robust augmented weighted Tchebycheff programs. A decision maker's most preferred solution is identified in the interactive robust weighted Tchebycheff procedure by progressively eliciting and incorporating the decision maker's preference information into the solution process. An example is presented to illustrate the solution approach and performance. The developed approach can also be applied to general multiobjective mixed integer linear programming problems.

Keywords: **Multiple objective programming, Robust optimization, Imprecise information, Portfolio selection, Interactive procedures.**

JEL Classification: C02, C61

1. Introduction

In today's fast paced and highly competitive economy, engaging in meaningful Research and Development (R&D) activities is essential for any organization striving to achieve and maintain competiveness. R&D projects are resource intensive and, therefore, benefits gained from and costs associated with each R&D project must be carefully considered. R&D project selection is a complex, non-trivial problem with important organizational implications.

Organizations usually have more candidate R&D projects than they have resources to support them. The purpose of R&D project portfolio selection is to select a feasible subset of promising projects as a portfolio from a set of candidate projects based on multiple criteria. R&D project portfolio selection

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is always constrained by limited resources such as budget, research staff, laboratory space, and other technical scarcities. In addition, R&D project portfolio selection may have other restrictions such as corporate policies and contractual relationships with other stakeholders. Furthermore, uncertainties are always involved in R&D, such as uncertainties in the outcomes of the projects, in the resource availability and usage, and in the interdependence and interactions among the projects. Given these constraints and uncertainties, R&D managers must select a portfolio of projects based on multiple criteria representing corporate goals or objectives. Objectives, such as profit maximization, market share maximization, risk minimization, or human resource utilization minimization, are usually conflicting and fraught with uncertainties which further complicate the R&D project portfolio selection. The challenge is how to select the best portfolio of R&D projects based on these competing objectives within the resource restrictions while giving consideration to uncertainties.

R&D is often an original endeavor with long lead time and unclear life time expenditure, resource usage and market outcome. These unique characteristics imply that much of the information required in making R&D decisions is very imprecise and impossible to accurately estimate. To address uncertainties, probabilistic and fuzzy approaches have been proposed to capture the imprecision of data by considering reasonable distributions to describe possible values of imprecise coefficients in optimization models. One drawback of such approaches is, however, that they cannot handle the situation where there is a possible range for each of these coefficients, but the most probable or plausible value within the range cannot be estimated (Carlsson *et al*. 2007). This calls for novel approaches which can more adequately capture the real world situation of R&D project portfolio selection.

The focus of this study is to develop a method for dealing with imprecise information associated with the multiobjective problem of selecting a portfolio of R&D projects. The proposed method integrates two complementary approaches to deal with both uncertainties and multiple objectives. Uncertainties in the problem coefficients, both in the objective functions and constraints, are modeled through robust optimization while the multiobjective problem is solved through interactive multiobjective programming. Interval uncertainties, *i.e.*, each imprecise coefficient belongs to an interval of real numbers without prior distribution details, are assumed. An interactive approach is used to capture the decision maker (DM)'s preference information with respect to the multiple objectives in the problem.

The remainder of the paper is organized as follows. The relevant R&D project portfolio selection literature is reviewed in Section 2. The nominal multiobjective programming model for R&D project portfolio selection is presented in Section 3. Following a brief introduction to robust optimization, the robust counterpart of the nominal multiobjective programming model is formulated in Section 4. The linear counterpart of the robust formulation is solved within an interactive procedure through the solution of augmented weighted Tchebycheff programs in Section 5. An example of a R&D project portfolio selection problem is presented to illustrate the proposed approach and the results of computational experimentation are reported in Section 6. Finally, the article concludes with a summary in Section 7.

2. Previous Work

Portfolio selection, whether financial, investment or R&D, is always fraught with uncertainty and is inherently multiobjective in nature. Steuer *et al.* (2005) presented a list of possible objectives in a financial portfolio selection problem. Because the objectives are incommensurate, the DM's preference information has to be used to make tradeoffs in order to find a final portfolio. However, a review of the literature in this area indicates that most of proposed solution techniques either focus on the multiple objectives or address the uncertainties but not both.

Multiobjective optimization solution techniques in solving the multiobjective project portfolio selection problem, like in other applications, can be classified into three major categories based on the time the DM's preference information is articulated, *i.e.*, those requiring *a priori* articulation of preference information, those requiring *a posteriori* articulation of preference information, and those requiring progressive articulation of preference information (Hwang and Masud 1979).

In the first category, *a priori* preference information articulation from the DM regarding the criteria is assumed, and a compromise solution is obtained by converting multiple objectives of the problem to a single objective. To this end, some authors assign different weights to the objective functions according to their importance to the DM, and use a weighted sum of the objective functions as a single objective function (Ghasemzadeh *et al*. 1999; Klapka and Piños 2002; Medaglia *et al*. 2008). This approach can only find basic solutions for linear problems and may fail to balance objective functions in relation to their importance (Steuer 1986). Some authors use goal programming to address this problem (Graves and Ringuest 1992; Schniederjans and Santhanam 1993; Zanakis *et al*. 1995; Santhanam and Kyparisis 1995; Badri *et al*. 2001; Lee and Kim 2001). Azmi and Tamiz (2010) provided a review of the goal programming approaches. However, setting aspiration levels and weights for the goals is challenging and may even result in a dominated solution (Santhanam and Kyparisis 1995; Ringuest and Graves 1989).

The second category includes approaches requiring *a posteriori* articulation of DM's preference information. Accordingly, it is assumed that *a priori* preference information articulation regarding the criteria is unavailable. Therefore, a two-phase procedure is implemented that first identifies the whole or a large set of efficient *i.e*., Pareto-optimal or nondominated, portfolios possibly using metaheuristics (Doerner *et al*. 2004, 2006; Ghorbani and Rabbani 2009; Rabbani *et al*. 2010; Stummer and Sun 2005; Carazo *et al*. 2010; Yu *et al*. 2012), and then explores the set of identified efficient solutions possibly through an interactive approach (Stummer and Heidenberger 2001, 2003). However, determining the set of all efficient solutions is challenging and becomes increasingly demanding or even impossible as the number of projects and/or the number of objectives grows because integer programming problems are usually NP hard. In addition, the DM may be confronted with a large number of competing portfolios in the second phase and selecting the one that is most preferred is not an easy task, which further complicates the process of project portfolio selection.

The third category includes interactive approaches in which the DM's preference information is articulated progressively during, and incorporated into, the solution process so as to locate the DM's most preferred solution. Interactive methods are the most promising approaches for solving multiobjective programming problems (Steuer 1986). Zopounidis *et al.* (1998) developed a multiple objective linear programming model to select a portfolio of stocks and used an interactive approach to solve the problem. Stummer and Heidenberger (2001, 2003) presented an interactive procedure for solving the multiobjective R&D project portfolio selection problems. Steuer *et al.* (2005) discussed tools and techniques from multiple criteria optimization to analyze and solve the portfolio selection problem.

The majority of project portfolio selection formulations in the literature are based on deterministic data. However, as mentioned earlier, an important characteristic of R&D project portfolio selection is that future attributes of R&D projects, *e.g.*, costs and revenues, availability and usage of human resources and material supplies, development of technical skills and risks, and market outcomes, are very difficult to estimate. Consequently, stochastic (Abdelaziz *et al.* 2007; Birge and Louveaux 1977; Gabriel *et al*. 2006; Medaglia *et al*. 2007; Gutjahr and Reiter 2010) and fuzzy (Aryanezhad *et al.* 2011; Bhattacharyya and Kar 2011; Coffin and Taylor 1996; Tolga 2008; Łapuńka 2012) approaches are introduced to the classical multiobjective programming formulations to address the issue of incomplete and imprecise information. However, both of these approaches assume prior details about coefficient distributions, an assumption which is often flawed for R&D projects as ground-breaking endeavors.

There are a few studies that address uncertainties of project portfolio selection within an interactive procedure. Nowak (2006) developed an interactive procedure for selecting one project that is based on STEM (Benayoun *et al.* 1971), a well-known interactive procedure for multiobjective programming. In this procedure, risk and uncertainty are modeled through stochastic dominance. Shing and Nagasawa (1999) proposed an interactive portfolio selection method for selecting a preferred portfolio from a set of candidate portfolios for the case where the mean and the variance of returns of securities have several scenarios with known occurrence probabilities.

Robust optimization is a relatively new approach which incorporates imprecise information by way of set inclusion, *i.e.*, the true value of a coefficient is contained in an interval characterized by the DM without any assumption on the distribution of the coefficients. Robust optimization addresses the problem of data uncertainty by guaranteeing the feasibility and optimality of the solution for the worst instances of the problem. As it is naturally a worst case approach, feasibility often comes at the cost of performance and the solutions obtained are usually overly conservative (see, *e.g.*, Soyster 1973). Recognizing this drawback, Bertsimas and Sim (2004) developed “the budget of uncertainty” approach to control the cumulative conservativeness of uncertain coefficients in the optimization problems.

Robust optimization has been applied to portfolio selection in different ways. Laguna (1995) applied robust optimization to a project selection problem. Since the large size of the discrete robust optimization formulations makes the use of classical optimization techniques impractical, a heuristic procedure was developed based on a probabilistic sampling approach. Liesiö (2006) used a robust approach to model the R&D project portfolio selection problem with multiple objectives when the DM’s preference information and the project data are incomplete. Projects included in all nondominated portfolios, called core projects, are identified while those not included in any nondominated portfolios, called exterior projects, are discarded. The final portfolio is selected by further analyzing the borderline projects. Düzgün and Thiele (2010) used robust optimization for single objective R&D project portfolio selection when the uncertainty in an objective coefficient is described using multiple ranges. Modarres and Hassanzadeh (2009) applied robust optimization to a single objective R&D project portfolio selection problem where maximization of the expected value of the R&D project portfolio subjecting to capital and resource constraints is considered.

The current study develops an interactive robust optimization procedure to solve multiobjective R&D portfolio selection problems with interval uncertainties in the coefficients of the objective functions and constraints. The approach developed in this study is the first method that uses robust optimization in an interactive procedure and opens a new avenue for solving multiobjective R&D portfolio selection problems with uncertainties. The methodology can also be directly used to tackle other, such as financial and investment, multiobjective portfolio selection problems.

3. Problem Statement

The aim of the multiobjective R&D project portfolio selection problem is to select a subset as a portfolio from a large set of possible candidate projects considering multiple conflicting objectives, subjected to a set of constraints. Let K denote the number of objective functions, m the number of constraints, and n the number of candidate projects in the entire set. There is no prior requirement for the number of projects to be selected into the portfolio. Without loss of generality, all objective functions are assumed to be minimized. The multiobjective R&D project portfolio selection model is stated as in (1) in the following

$$\begin{aligned} \min \quad & z_k = f_k(\mathbf{x}) && \forall k \\ \text{s.t.} \quad & g_i(\mathbf{x}) \le b_i && \forall i \\ & x_j \in \{0,1\} && \forall j\,. \end{aligned} \tag{1}$$

In the model, $\mathbf{x} \in \Re^n$ is the vector of binary decision variables, $f_k(\mathbf{x}) = \sum_{j=1}^{n} c_{kj} x_j$ is the k th objective function, and $g_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j \le b_i$ is the i th constraint. Although each application is different, the objective functions may include the maximization of total expected profit, maximization of expected market share, or minimization of total expected risk, while the constraints may include limited budget, scarce human and material resources, and interdependence and interaction among the candidate projects. A project j is selected into the portfolio if $x_j = 1$ and otherwise if $x_j = 0$. In addition to the binary decision variables representing the selection of projects, other decision variables may be used in the model to represent the interdependencies and interactions among the projects in some specific applications. The multiobjective R&D project portfolio selection model (1) is the nominal model assuming the values of a_{ij}, $\forall i, j$, and c_{kj}, $\forall k, j$, are exactly known. The nominal model (1) is an ordinary multiobjective binary integer linear programming model.

Since the objective functions are usually in conflict, model (1) usually does not have a single feasible solution that simultaneously minimizes all K objective functions. The optimal solution is defined to be a feasible solution that maximizes the DM's value function (Steuer 1986; Yu 1985). Because the DM's value function is not readily available, the solution process of the multiobjective R&D project portfolio selection model (1) is to search for a solution which is most preferred by the DM.

The following concepts are borrowed from Steuer (1986). The set of solutions satisfying all constraints, *i.e.*, $X = \{\mathbf{x} \in \Re^n \mid g_i(\mathbf{x}) \le b_i, \forall i, x_j \in \{0,1\}, \forall j\}$, is the feasible region, and a point $\mathbf{x} \in X$ is a feasible solution, in decision space. The set $Z = \{\mathbf{z} \in \Re^K \mid z_k = f_k(\mathbf{x}) \mid \mathbf{x} \in X\}$ is the feasible region in criterion space. A point $\mathbf{z} \in Z$ is a feasible solution in criterion space or a feasible criterion vector. A point $\bar{\mathbf{z}} \in Z$ is a nondominated criterion vector if there does not exist any criterion vector $\mathbf{z} \in Z$, such that $\mathbf{z} \le \bar{\mathbf{z}}$ and $\mathbf{z} \ne \bar{\mathbf{z}}$. $\bar{Z}$ is used to represent the set of all nondominated solutions in criterion space. A point $\bar{\mathbf{x}} \in X$ is an efficient solution in decision space if $\bar{\mathbf{z}} \in \bar{Z}$ such that $\bar{z}_k = f_k(\bar{\mathbf{x}})$, $\forall k$. $\bar{X}$ is used to represent the set of all efficient solutions in decision space. A criterion vector $\hat{\mathbf{z}} \in Z$ is optimal if it maximizes the DM's value function. However, a DM's value function in real life problems is hard to estimate and its functional form is usually unknown (Yu 1985). If $\hat{\mathbf{z}}$ is optimal, $\hat{\mathbf{z}} \in \bar{Z}$, *i.e.*, an optimal solution must be nondominated. A point $\mathbf{z}^* \in \Re^K$, such that $z_k^* = \min\{f_k(\mathbf{x}), \mathbf{x} \in X\}$, $\forall k$, is the ideal point. For most multiobjective programming problems, $\mathbf{z}^* \notin Z$, *i.e.*, $\mathbf{z}^*$ is infeasible. A point $\mathbf{x}^* \in \Re^n$, such that $z_k^* = f_k(\mathbf{x}^*)$, $\forall k$, usually does not exist (Sun 2005). A point $\mathbf{z}^{**} \in \Re^K$, such that $z_k^{**} = z_k^* - \varepsilon_k$, where $\varepsilon_k > 0$ and small, is called a utopian point.

When a multiobjective programming problem is solved, especially when an interactive procedure or an approach requiring *a posteriori* articulation of the DM's preference information is used, many nondominated solutions need to be generated as trial solutions. These nondominated solutions are usually evaluated by the DM so as to elicit preference information from the DM. Nondominated solutions are usually generated by solving augmented weighted Tchebycheff programs derived from the nominal multiobjective programming model (1) (Steuer 1986). The weighting vector space is defined as

$$\mathbf{W} = \{\mathbf{w} \in \Re^K \mid w_k > 0, \sum_{k=1}^{K} w_k = 1\}. \tag{2}$$

Any $\mathbf{w} \in \mathbf{W}$ is a weighting vector. For a given $\mathbf{w} \in \mathbf{W}$, an augmented weighted Tchebycheff program for the nominal R&D project portfolio selection model (1) is formulated as in (3) in the following

$$\begin{aligned}
\min \quad & \alpha + \rho \sum_{k=1}^{K} (z_k - z_k^{**}) & \\
\text{s.t.} \quad & \alpha \geq w_k (z_k - z_k^{**}) & \forall k \\
& z_k = f_k(\mathbf{x}) & \forall k \\
& g_i(\mathbf{x}) \leq b_i & \forall i \\
& x_j \in \{0,1\} & \forall j \\
& z_k \text{ unrestricted} & \forall k \\
& \alpha \geq 0, &
\end{aligned} \tag{3}$$

where $\rho > 0$ is a small scalar. Usually $\rho = 0.001$ is sufficient.

The augmented weighted Tchebycheff program (3) is a single objective binary integer linear programming problem. If its optimal solution is represented by the composite vector $(\mathbf{x}_\mathbf{w}, \mathbf{z}_\mathbf{w}, \alpha_\mathbf{w})$, then $\mathbf{x}_\mathbf{w} \in \bar{X}$ and $\mathbf{z}_\mathbf{w} \in \bar{Z}$, *i.e.*, $\mathbf{x}_\mathbf{w}$ is efficient and $\mathbf{z}_\mathbf{w}$ is nondominated. For a given $\mathbf{w} \in \mathbf{W}$, the augmented weighted Tchebycheff program (3) generates a given nondominated solution. By using a widely dispersed set of weighting vectors in $\mathbf{W}$, a widely dispersed set of representative nondominated solutions can be generated.

As previously stated, the values of most of the coefficients in the multiobjective R&D project portfolio selection model (1) are not known with certainty given the nature of the R&D project portfolio selection problem. When the values of these coefficients are not precisely known, the solution obtained for the nominal model (1) may not be close to the true most preferred solution of the DM or, even worse, could be infeasible for a realization of these uncertain coefficients. Given that a_{ij}, $\forall i, j$, and c_{kj}, $\forall k, j$, are uncertain and their exact values are unknown but within a certain interval, the focus of this study is on finding a solution of model (1) such that the solution is not only feasible with a very high probability, but also is very close to the most preferred solution of the DM. An interactive robust weighted Tchebycheff procedure is proposed for this purpose.

4. Robust Optimization for R&D Project Portfolio Selection

A robust optimization framework is briefly discussed first for single objective optimization problems. This framework is then extended to multiple objective optimization problems through the augmented weighted Tchebycheff program (3).

4.1 The robust optimization framework for single objective problems

Consider the following standard linear programming problem

$$\begin{array}{lll} \min & f(\mathbf{x}) & \\ \text{s.t.} & g_i(\mathbf{x}) \le b_i & \forall i \\ & \mathbf{l} \le \mathbf{x} \le \mathbf{u} & \end{array} \tag{4}$$

where $g_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j \le b_i$, as in (1), is the i th constraint and $f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j$ is the single objective function of the problem. The standard linear programming model in (4) with precise a_{ij} , $\forall i, j$, is the nominal formulation.

Now assume that each a_{ij} is an uncertain coefficient with unknown exact value in the interval $[\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ where $\bar{a}_{ij}$ is the nominal value and $\hat{a}_{ij}$ is the half-interval width of a_{ij} . The precise values of c_j , $\forall j$, are assumed to be known. The purpose of robust optimization is to find an optimal solution, called the robust solution, which remains feasible for almost all possible realizations of the uncertain problem coefficients. Obviously, as much as it is unlikely that all uncertain coefficients are equal to their nominal values, it is also unlikely that they are all equal to their worst-case values. The worst-case solution actually occurs with a negligible probability because large deviations in the coefficients a_{ij} tend to cancel out each other as n grows. Consequently, the most conservative approach, where all coefficients are equal to their worst-case values, leads to a severe deterioration of the optimal solution without being fairly justified in practice. Therefore, adjusting the degree of conservatism of the solution in order to make a reasonable trade-off between robustness and performance is a necessity (Bertsimas and Sim 2004).

This concept is quantified by reformulating the nominal model in (4). The absolute value of the scaled deviation from its nominal value of the uncertain coefficient a_{ij} , denoted by δ_{ij} , is defined as

$$\delta_{ij} = |(a_{ij} - \bar{a}_{ij}) / \hat{a}_{ij}| \qquad \forall i, j \, . \tag{5}$$

Apparently, δ_{ij} takes values in the interval $[0,1]$. A budget of uncertainty Γ_i is imposed to the i th constraint in the following sense

$$\sum_{j=1}^{n} \delta_{ij} \le \Gamma_i \qquad 0 \le \Gamma_i \le n, \tag{6}$$

where $\Gamma_i = 0$ and $\Gamma_i = n$ correspond to the nominal and worst cases, respectively. Bertsimas and Sim (2004) showed that letting the budget of uncertainty Γ_i vary in the interval $[0,n]$ makes it possible to build a model where performance is appropriately adjusted against robustness. Intuitively, the use of Γ_i can rule out large deviations in $\sum_{j=1}^{n} a_{ij} x_j$ that play a predominant role in worst-case analysis. When each a_{ij} is treated as a variable, the nonlinear robust formulation of the nominal model in (4) can be stated as

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \max_{\mathbf{a}_i} [\bar{g}_i(\mathbf{a}_i, \mathbf{x}) \mid \sum_{j=1}^{n} \delta_{ij} \le \Gamma_i] \le b_i \qquad \forall i \\ & \mathbf{l} \le \mathbf{x} \le \mathbf{u}. \end{aligned} \tag{7}$$

where $\mathbf{a}_i$ is the vector of uncertain coefficients in the i th constraint with each $a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$ and $\bar{g}_i(\mathbf{a}_i, \mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j$ is the counterpart of $g_i(\mathbf{x})$ in (4) but with each a_{ij} treated as a variable. Bertsimas and Sim (2004) proved that the nonlinear robust formulation in (7) has the following robust linear counterpart

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \bar{a}_{ij} x_j + \Gamma_i q_i + \sum_{j=1}^{n} r_{ij} \le b_i && \forall i \\ & q_i + r_{ij} \ge \hat{a}_{ij} y_j && \forall i, j \\ & -y_j \le x_j \le y_j,\ l_j \le x_j \le u_j && \forall j \\ & q_i \ge 0 && \forall i \\ & y_j \ge 0 && \forall j \\ & r_{ij} \ge 0 && \forall i, j \end{aligned} \tag{8}$$

A highly attractive feature of this formulation is that this linear counterpart is of the same class as the nominal model in (4) which can be easily solved with standard optimization techniques. Moreover, if some of the variables of the original model are constrained to be integers, they then retain the same integer type in the robust counterpart (8). Finally, Bertsimas and Sim (2004) showed that even if the budget of uncertainty constraints are not satisfied, the robust solution will remain feasible with a very high probability.

4.2 Application to multiple objective problems

Under uncertainty, the problem coefficients in (1) are uncertain and, hence, the selected portfolio must be robust, *i.e.*, the solution should remain feasible (constraint robust), efficient and most preferred by the DM (objective function robust) under almost all possible realizations of uncertain coefficients. Both a_{ij} and c_{kj} are considered uncertain and their uncertainty is captured using the interval uncertainty discussed earlier. The nominal value $\overline{a}_{ij}$ and the half-interval width $\hat{a}_{ij}$ of each a_{ij} are the same as in the single objective optimization problem discussed above. The nominal value and the half-interval width of c_{kj} are represented by $\overline{c}_{kj}$ and $\hat{c}_{kj}$, respectively. The k th uncertain objective function is expressed as $\overline{f}_k(\mathbf{c}_k, \mathbf{x}) = \sum_{j=1}^{n} c_{kj} x_j$ where $\mathbf{c}_k$ is the vector of uncertain coefficients in the k th objective function with each $c_{kj} \in [\overline{c}_{kj} - \hat{c}_{kj}, \overline{c}_{kj} + \hat{c}_{kj}]$. While being the counterpart of $f_k(\mathbf{x})$ in (1), $\overline{f}_k(\mathbf{c}_k, \mathbf{x})$ is a function of both $\mathbf{c}_k$ and $\mathbf{x}$, because each c_{kj} is treated as a variable. Similar to δ_{ij} defined for a_{ij} in (5), the absolute value of scaled deviation δ'_{kj} of c_{kj} from its nominal value $\overline{c}_{kj}$ is defined as

$$\delta'_{kj} = |(c_{kj} - \overline{c}_{kj}) / \hat{c}_{kj}| \qquad \forall k, j \,. \tag{9}$$

Similar to (6), a budget of uncertainty Γ'_k is imposed to the k th objective function such that

$$\sum_{j=1}^{n} \delta'_{kj} \le \Gamma'_k \qquad 0 \le \Gamma'_k \le n \,, \tag{10}$$

where $\Gamma'_k = 0$ and $\Gamma'_k = n$ correspond to the nominal and worst cases, respectively. Note that while Γ_i controls the robustness of the i th constraint, Γ'_k controls the robustness of the k th objective function against the level of conservatism. For notational convenience, let $\mathbf{\Gamma} \in \Re^m$ and $\mathbf{\Gamma}' \in \Re^K$ be the vectors of budgets of uncertainty for the constraints and for the objective functions, respectively. Imposing the budgets of uncertainty for the constraints and the objective functions will ensure that the solution will remain both constraint robust and objective function robust. The nonlinear robust formulation of the nominal multiobjective programming model in (1) is stated as

$$\begin{aligned} \min \quad & z_k = \max_{\mathbf{c}_k} \left[\overline{f}_k(\mathbf{c}_k, \mathbf{x}) \,|\, \sum_{j=1}^{n} \delta'_{kj} \le \Gamma'_k \right] && \forall k \\ \text{s.t.} \quad & \max_{\mathbf{a}_i} \left[\overline{g}_i(\mathbf{a}_i, \mathbf{x}) \,|\, \sum_{j=1}^{n} \delta_{ij} \le \Gamma_i \right] \le b_i && \forall i \\ & x_j \in \{0,1\} && \forall j \,. \end{aligned} \tag{11}$$

Unlike the single objective model (7), each c_{kj} in the objective functions is also considered uncertain in (11).

Any feasible solution to the above model is called a robust feasible solution. The set of all robust feasible solutions, *i.e.*, $X^{\Gamma} = \{\mathbf{x} \in \Re^n \mid \max_{\mathbf{a}_i}[\bar{g}_i(\mathbf{a}_i, \mathbf{x}) \mid \sum_{j=1}^{n} \delta_{ij} \le \Gamma_i] \le b_i \ \forall i, x_j \in (0,1)\}$, is called the robust feasible region in decision space for a given vector $\mathbf{\Gamma}$. A $\mathbf{x} \in X^{\Gamma}$ is called a robust feasible solution in decision space. The set $Z^{\Gamma,\Gamma'} = \{\mathbf{z} \in \Re^K \mid z_k = \max_{\mathbf{c}_k}[\bar{f}_k(\mathbf{c}_k, \mathbf{x}) \mid \sum_{j=1}^{n} \delta'_{kj} \le \Gamma'_k], \mathbf{x} \in X^{\Gamma}\}$ is the robust feasible region in criterion space for the given vectors $\mathbf{\Gamma}'$ and $\mathbf{\Gamma}$. A $\mathbf{z} \in Z^{\Gamma,\Gamma'}$ is called a robust feasible solution in criterion space or a robust criterion vector. A nondominated robust criterion vector $\bar{\mathbf{z}} \in Z^{\Gamma,\Gamma'}$, an efficient robust solution $\bar{\mathbf{x}} \in X^{\Gamma}$, and an optimal robust solution $\hat{\mathbf{z}} \in Z^{\Gamma,\Gamma'}$ can be defined similarly to their counterparts for the nominal model (1). The robust ideal point $\mathbf{z}^* \in \Re^K$ is defined such that $z_k^* = \min_{\mathbf{x}}\{\max_{\mathbf{c}_k}[\bar{f}_k(\mathbf{c}_k, \mathbf{x}) \mid \sum_{j=1}^{n} \delta'_{kj} \le \Gamma'_k], \mathbf{x} \in X^{\Gamma}\}$. A robust utopian point is also defined as $\mathbf{z}^{**} \in \Re^K$ such that $z_k^{**} = z_k^* - \varepsilon_k$ with $\varepsilon_k > 0$ and small.

For a given weighting vector $\mathbf{w} \in \mathbf{W}$, a robust augmented weighted Tchebycheff program for the nonlinear programming model in (11) is formulated from (3) as the following

$$
\begin{array}{lll}
\min & \alpha + \rho \sum_{k=1}^{K} (z_k - z_k^{**}) & \\
\text{s.t.} & \alpha \ge w_k (z_k - z_k^{**}) & \forall k \\
& z_k = \max_{\mathbf{c}_k}[\bar{f}_k(\mathbf{c}_k, \mathbf{x}) \mid \sum_{j=1}^{n} \delta'_{kj} \le \Gamma'_k] & \forall k \\
& \max_{\mathbf{a}_i}[\bar{g}_i(\mathbf{a}_i, \mathbf{x}) \mid \sum_{j=1}^{n} \delta_{ij} \le \Gamma_i] \le b_i & \forall i \\
& x_j \in \{0,1\} & \forall j \\
& z_k \text{ unrestricted} & \forall k \\
& \alpha \ge 0. &
\end{array}
\tag{12}
$$

Similar to the coefficients in the objective function of model (7), the coefficients in the objective function of model (12) are exactly known.

An optimal solution to (12) minimizes the augmented weighted Tchebycheff metric between $\mathbf{z}^{**}$ and any $\mathbf{z} \in Z^{\Gamma,\Gamma'}$ while respecting the budget of uncertainty constraints. The solution to this formulation has some interesting properties. First, it is a nondominated solution for the selected $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$. Second, unlike its nominal counterpart, it is robust, *i.e.*, insensitive to existing uncertainties in both the objective functions and constraints. This means that given all possible realizations of a_{ij} and c_{kj}, the solution of (12) not only will have a much higher probability of being feasible than the nominal solution of (3) but also will have a corresponding criterion vector which performs comparably well to the nondominated nominal criterion vector. These properties are significant because model (12) can assist the DM as a tool in finding nondominated robust solutions by properly balancing performance versus robustness. Using this formulation, the nominal solution closest to the nominal utopian point $\mathbf{z}^{**}$, measured by the augmented weighted Tchebycheff metric, is slightly sacrificed but, in return, this sacrifice is compensated by the robustness of the solution.

Proposition 1. Model (12) has the following mixed binary integer linear programming counterpart

$$
\begin{array}{lll}
\min & d & \\
\text{s.t.} & \alpha + \rho \sum_{k=1}^{K} \alpha_k - d \le 0 & \\
& w_k \alpha_k - \alpha \le 0 & \forall k \\
& \sum_{j=1}^{n} \bar{c}_{kj} x_j + \Gamma'_k q'_k + \sum_{j=1}^{n} r'_{kj} - \alpha_k \le z_k^{**} & \forall k \\
& \sum_{j=1}^{n} \bar{a}_{ij} x_j + \Gamma_i q_i + \sum_{j=1}^{n} r_{ij} \le b_i & \forall i \\
& q'_k + r'_{kj} \ge \hat{c}_{kj} y_j & \forall k, j \\
& q_i + r_{ij} \ge \hat{a}_{ij} y_j & \forall i, j \\
& x_j \le y_j ,\ y_j \ge 0 ,\ x_j \in \{0,1\} & \forall j \\
& q_i \ge 0 & \forall i \\
& \alpha_k , q'_k \ge 0 & \forall k \\
& r'_{kj} \ge 0 & \forall k, j \\
& r_{ij} \ge 0 & \forall i, j \\
& d, \alpha \ge 0 . &
\end{array}
\qquad (13)
$$

Proof. See Appendix B.

5. The Interactive Robust Weighted Tchebycheff Procedure

In recent years, many interactive methods have been proposed to solve the nominal model (1) of the multiobjective project portfolio selection problem. In this study, the interactive weighted Tchebycheff procedure (Steuer and Choo 1983; Steuer 1986) is used as a framework to solve the robust version (11) of the multiobjective project portfolio selection problem. The interactive weighted Tchebycheff procedure is a weighting vector space reduction method (Steuer 1986) in which the weighting vectors are generated from progressively reduced subsets of $\mathbf{W}$ defined in (2).

The mixed binary integer linear programming counterpart (13) of the robust augmented weighted Tchebycheff program (12) is used to generate nondominated robust solutions. Within the interactive weighted Tchebycheff procedure, a set of dispersed weighting vectors is generated at each iteration. The set is then filtered to reduce to a smaller manageable subset. The mixed binary integer linear programming model in (13) is then solved for each weighting vector in this smaller subset to obtain a set of nondominated criterion vectors. The resulting nondominated robust criterion vectors are then filtered to obtain a smaller subset of dispersed ones. This subset is presented to the DM who selects the most preferred robust solution. In the next iteration, the weighting vector space is reduced around the weighting vector corresponding to the current most preferred solution selected by the DM, new dispersed weighting vectors are generated in this reduced weighting vector space, the newly generated set of weighting vectors is filtered, new nondominated robust solutions are generated, and so on. The procedure terminates after a predetermined number of iterations have been performed or when the DM is satisfied with a nondominated robust solution that has already been found.

In the following, the integer I represents the iteration number. While indicating superscripts in $W^{(I)}$, $w^{(I)}$ and $\mathbf{z}^{(I)}$, I denotes power in r^I. The interactive robust weighted Tchebycheff procedure (Steuer and Choo 1983; Steuer 1986) described step-by-step in the following is similar to that in Drinka *et al*. (1996).

Step 1. Determine the maximum number of iterations $I^{\max}$, the number of solutions P to be presented to the DM at each iteration, and the weighting vector space reduction factor r. Let $I=0$ and $[l_k, u_k]=[0,1]$ for all k. Obtain z_k^* by solving the robust model (A.1) for each k and then determine the robust utopian point $\mathbf{z}^{**}$.

Step 2. Let $I=I+1$. From $W^{(I)}=\{w\in\Re^K \mid w_k\in(l_k,u_k),\sum_k w_k=1\}$ randomly generate $20K$ weighting vectors. Reduce the $20K$ weighting vectors to obtain the $2P$ most widely dispersed ones.

Step 3. Solve the robust mixed binary integer linear programming model (13) for each weighting vector to obtain $2P$ nondominated robust criterion vectors. Reduce the $2P$ nondominated robust criterion vectors to P most dispersed ones.

Step 4. Present the P nondominated robust criterion vectors, together with the most preferred solution from the previous iteration $\mathbf{z}^{(I-1)}$ if $I>1$ to the DM to acquire the most preferred solution $\mathbf{z}^{(I)}$ at iteration I.

Step 5. Terminate the solution process if $I=I^{\max}$ or if the DM is satisfied with $\mathbf{z}^{(I)}$; continue otherwise.

Step 6. Compute the weighting vector $\mathbf{w}^{(I)}$ that can generate the current most preferred solution $\mathbf{z}^{(I)}$ with

$$w_k^{(I)}=\frac{(z_k^{(I)}-z_k^{**})^{-1}}{\sum_{k'=1}^{K}(z_{k'}^{(I)}-z_{k'}^{**})^{-1}}, \tag{14}$$

and update l_k and u_k using (15) in the following

$$[l_k, u_k] = \begin{cases} [0, r^I], & w_k^{(I)} - r^I/2 \le 0 \\ [1 - r^I, 1], & w_k^{(I)} - r^I/2 \ge 1 \\ [w_k^{(I)} - r^I/2, w_k^{(I)} + r^I/2], & \text{Otherwise.} \end{cases} \tag{15}$$

Go to Step 2.

Note that when $\Gamma = 0$ and $\Gamma' = 0$, the above procedure reduces to the interactive weighted Tchebycheff procedure for the nominal problem (1) . The $20K$ weighting vectors in Step 2 is a generally accepted size in order to generate widely dispersed weighting vectors (Steuer 1986). Similar to Steuer (2003), the relative distance measure is used to reduce the $20K$ weighting vectors to $2P$ in Step 2, and to reduce the $2P$ robust criterion vectors to P in Step 3. Appendix C has a discussion on the approach used to reduce the set of vectors.

The choices of $I^{\max}$ and P are dependent on DM's preferences. Larger values for $I^{\max}$ increase the decision making time and the burden on the DM but increase solution quality, whereas larger values of P make the comparison of multiple solutions more time consuming and increase the burden on the DM but may elicit more preference information from the DM. Both $I^{\max}$ and P may be revised during the solution process upon DM's desire. For a detailed discussion of the interactive weighted Tchebycheff procedure, see Steuer and Choo (1983) and Steuer (1986).

In practice, a variation to the interactive robust weighted Tchebycheff procedure is the combined Tchebycheff/reference point interactive multiobjective programming procedure (Steuer *et al.* 1993). This variation is not convenient in performing a computational experiment but may more quickly lead to a solution satisfied by the DM. In this variation, a reference point or an aspiration vector $\mathbf{z}^{(a)} \in \Re^K$ specified by the DM instead of the current most preferred solution $\mathbf{z}^{(I)}$ is used to compute $\mathbf{w}^{(I)}$ in (14) in Step 6. The reference point $\mathbf{z}^{(a)}$ is not in the set of criterion vectors presented to the DM but the DM specifies this reference point as a desired solution. The weighting vector $\mathbf{w}^{(I)}$ is then computed with (16) in the following

$$w_k^{(I)} = \frac{(z_k^{(a)} - z_k^{**})^{-1}}{\sum_{k'=1}^{K} (z_{k'}^{(a)} - z_{k'}^{**})^{-1}} . \tag{16}$$

The aspiration vector $\mathbf{z}^{(a)}$ specified by the DM may be dominated or infeasible. However, when $\mathbf{z}^{(a)}$ replaces $\mathbf{z}^{**}$, the augmented weighted Tchebycheff program projects $\mathbf{z}^{(a)}$ to the nondominated set $\bar{Z}$ or $Z^{\Gamma,\Gamma'}$. Note that the variable $\alpha \ge 0$ in (12) and (13) becomes unrestricted in sign when $\mathbf{z}^{(a)}$ replaces $\mathbf{z}^{**}$ in the augmented weighted Tchebycheff program.

Value functions are usually used as proxy DMs in computational experiments to test the performance of solution procedures. L_ρ-metric value functions are used to act as proxy DMs in this study in order to simulate the solution process using the interactive robust weighted Tchebycheff procedure with the involvement of a DM. The L_ρ -metric value function of a criterion vector $\mathbf{z} \in \Re^K$ has the following functional form

$$V_\rho(\mathbf{z}) = \kappa - \left[\sum_{k=1}^{K} [w'_k (z_k - z_k^{**})]^\rho \right]^{\frac{1}{\rho}}, \tag{17}$$

where κ is a large constant ensuring $V_\rho(\mathbf{z}) > 0$ for all feasible robust criterion vectors, $\mathbf{w}' \in \mathbf{W}$ is a weighting vector selected by the user for the purpose of computational experiment, and $\rho \geq 1$ is an integer. The value function $V_\rho(\mathbf{z})$ in (17), however, is used only as a proxy DM to evaluate representative solutions in Step 4 of the interactive robust weighted Tchebycheff procedure and is not used directly in searching for the optimal solution in the solution process.

6. An Illustrative Example

A project portfolio selection problem presented in Santhanam and Kyparisis (1995) is used to demonstrate the proposed interactive robust Tchebycheff procedure. Ringuest and Graves (2000) also used the same problem to test their solution method.

An IT company faces the selection of a portfolio from a total of $n = 14$ projects where data on costs, benefits, and other related information for these projects are estimated. Existing cost interdependencies and synergistic benefits among projects are also identified. There are $m = 2$ resource constraints for hardware costs and software costs of the projects that must be satisfied. The problem has $K = 3$ objectives: maximization of total benefits, minimization of total risk scores, and minimization of total miscellaneous costs. The total hardware and software budgets are 20,000 and 6000, respectively. Table 1 and Table 2 present the original problem data.

Insert Table 1 & Table 2 Here

Similar to those in Santhanam and Kyparisis (1995), 22 binary variables (x_j) are defined and used to model the selection of a portfolio from the $n = 14$ projects as well as to model the 8 project interdependencies. The final linearized multiobjective binary integer linear programming model is formulated as model (D.1) in Appendix D. Note that maximization of total benefits is treated as a minimization objective function in model (D.1). However, the corresponding positive values of this objective function are reported in the tables and texts. The ideal solution is $\mathbf{z}^* = (60643, 5, 0)$. For this illustrative example, $\varepsilon_k = 0$, $\forall k$, is used, hence $\mathbf{z}^{**} = \mathbf{z}^*$.

In the following, a large set of nondominated solutions is generated first for the deterministic, *i.e.*, the nominal, model by solving augmented weighted Tchebycheff programs (3). The interactive weighted Tchebycheff procedure is then applied to the nominal model. The interval uncertainties on coefficients in the objective functions and constraints are considered next and the interactive robust weighted Tchebycheff procedure is applied to find preferred robust solutions for proxy DMs represented by the L_p-metric value functions (17). A simulation study is finally performed to evaluate the feasibility and quality of the solutions by introducing uncertainties into the coefficients. The parameters used in the L_p-metric value function (17) are arbitrarily set to $\kappa = 20{,}000$ and $\mathbf{w}' = (0.3, 0.4, 0.3)$ for this example. In the interactive robust weighted Tchebycheff procedure, $I^{\max} = 8$, $P = 8$, and $r = 0.2$ are used. In the following tables reporting results, the headings S&K, R&G and H&N&S represent the results from Santhanam and Kyparisis (1995), Ringuest and Graves (2000), and the current study, respectively.

All computations were conducted on a personal computer with a 2 GHz Core 2 Duo processor and 3 GB of RAM. The reported results reflect the performance of the proposed approach on this computer.

6.1 Generation of nondominated solutions

The augmented weighted Tchebycheff program (3) formulated from model (D.1) is used to generate nondominated solutions for the nominal model (1). Generation of a large set of nondominated solutions is usually the first phase in solving multiobjective programming problems requiring *a posteriori* articulation of the DM's preference information (Hwang and Masud 1979). A set of 300 weighting vectors is randomly generated from $\mathbf{W}$ defined in (2) that is then reduced to a subset of 200 most dissimilar ones. An augmented weighted Tchebycheff program is solved for each of the 200 weighting vectors. Because some augmented weighted Tchebycheff programs formulated with different weighting vectors share the same optimal solution, only 39 distinct nondominated criterion vectors are obtained. The solution process took a total CPU time of 190 seconds.

Santhanam and Kyparisis (1995) solved this problem with a goal programming approach with preemptive priorities. They considered two different priorities with different goal targets and solved 25 problems which yielded 8 nondominated and 1 dominated solutions. Ringuest and Graves (2000) solved the same problem using the Parameter Space Investigation (PSI) method (Steuer and Sun 1995). Using the PSI method, they randomly generated binary solutions over a hyperrectangle that completely encloses the feasible region. These binary solutions were then evaluated by each of the constraints to check for feasibility. Those not satisfying any of the constraints were discarded until 100 feasible solutions were generated. The total number of solutions, including feasible and infeasible, is not reported. The solution set was then screened for dominance which resulted in 33 distinct solutions among which 31 are not dominated by others. Table 3 lists all solutions reported by Santhanam and Kyparisis (1995), Ringuest and Graves (2000), and those generated by solving augmented weighted Tchebycheff programs in this study.

Insert Table 3 Here

A complete enumeration (CE) was performed to find all efficient solutions for this problem. The problem has 234 feasible solutions of which 63 are efficient and 171 are not. From the set of efficient solutions, 54 distinct nondominated criterion vectors are found. There is a difference between the number of efficient solutions and the number of nondominated criterion vectors because some different portfolios share the same criterion vectors. Because projects 7 and 8 have the same coefficients in the objective functions, given that the solution remains feasible, replacing one with the other in an efficient portfolio will create a different efficient portfolio with the same criterion vector. A further examination of the 31 solutions not dominated by others reported in Ringuest and Graves (2000) found only 15 of them are actually nondominated.

The total number of feasible solutions, number of nondominated solutions and the number of nondominated criterion vectors found by these methods are summarized in Table 4. Table 4 shows that the augmented weighted Tchebycheff program can generate a richer set of nondominated criterion vectors than the other methods with similar computational efforts. In addition, solutions generated with the augmented weighted Tchebycheff programs are guaranteed to be nondominated whereas those found with the other two methods may be dominated.

Insert Table 4 Here

6.2 Interactive weighted Tchebycheff procedure applied to the nominal model

The interactive weighted Tchebycheff procedure is applied to the nominal model to search for a final solution for proxy DMs represented by the L_p -metric value functions (17). For the nominal model, $\Gamma'_k = 0$ for $k = 1, 2, 3$ and $\Gamma_i = 0$ for $i = 1, 2$.

Because all nondominated solutions are found through CE, the optimal solution for each proxy DM can be found by directly evaluating all nondominated solutions with the corresponding value function and then selecting the solution with the largest value. These optimal solutions are then used to measure the quality of the solutions obtained with different methods. Solutions obtained with the interactive weighted Tchebycheff procedure at successive iterations as well as the optimal solutions obtained by R&G, S&K, and CE are summarized in Table 5, Table 6 and Table 7 for the L_p -metric value functions with $p = 2$, $p = 4$ and $p = \infty$, respectively. As it is clear from these tables, the interactive weighted Tchebycheff procedure can find the optimal solution within 3 or 4 iterations. In addition, although Santhanam and Kyparisis (1995) and Ringuest and Graves (2000) did not use any value function as proxy DM to search for the optimal solution, their best solutions for the L_p -metric value functions with $p = 2$, $p = 4$ and $p = \infty$ are close to the optimal solutions.

Insert Table 5, Table 6 & Table 7 Here

6.3 Interactive robust weighted Tchebycheff procedure applied to the robust model

In the following, the data in Table 1 and Table 2 are viewed as nominal values and interval uncertainties are introduced into all costs, benefits, and risk scores. Table 8 and Table 9 present the half-interval widths for the data in Table 1 and Table 2, respectively.

Insert Table 8 & Table 9 Here

Given all possible outcomes of the uncertainties, the purpose of the solution process is to find a DM's most preferred nondominated solution which is robust in terms of uncertainties in both constraints and objective functions. In the solution process, the mixed binary integer linear programming counterpart (13) of the robust augmented weighted Tchebycheff program (12) is solved in Step 3 of the iterative robust weighted Tchebycheff procedure. However, in order for the DM to be impartial toward robustness of the presented solutions, the proxy DM is assumed to always choose the preferred solution in Step 4 using the nominal criterion vectors, *i.e.*, the criterion vectors computed with the nominal coefficient values in the objective functions are evaluated with the L_p -metric value functions (17). For demonstration purposes, arbitrary values of $\Gamma'_k = 0.7$ for all $k = 1, 2, 3$ and $\Gamma_i = 0.5, 1.0, 1.5$ for all $i = 1, 2$ are used in order to examine the effect of the budget of uncertainty levels on solution feasibility and quality. To keep the results concise, only the final solutions obtained with the iterative robust weighted Tchebycheff procedure are reported. The solutions for the L_p -metric value functions with $p = 2$, $p = 4$ and $p = \infty$ are reported in Table 10, Table 11 and Table 12, respectively.

Insert Table 10, Table 11 & Table 12 Here

As expected, the value function is negatively correlated with the budget of uncertainty levels because an increase in the budget of uncertainty generally shrinks the feasible region and may render the current solution infeasible. It is also observed that as consequences of choosing an optimal robust solution instead of the optimal nominal solution, the value function deteriorates by at most 0.49%, 0.80%, and 1.45% for the L_p -metric value functions with $p = 2$, $p = 4$ and $p = \infty$, respectively. Therefore, it appears that for each p , the optimal robust solution is at the close proximity of the optimal nominal solution.

6.4 Simulation study

Since input data are fraught with uncertainty, comparing the performance of solutions using the nominal criterion vectors may not legitimately capture the effects of uncertainty on solution feasibility and quality.

To address this issue, a simulation study is performed to mimic the performance of the above nominal and robust solutions in the real world.

For each uncertain coefficient in the objective functions and in the constraints of the uncertain project portfolio selection problem, a value is randomly selected from its uncertainty interval. These selected coefficient values, instead of the nominal values, are then used in model (D.1) to formulate the project portfolio selection model. The formulated model is not solved but is used to evaluate the final solutions of the nominal model and of the robust model obtained with the interactive robust weighted Tchebycheff procedure. The best solutions reported in Santhanam and Kyparisis (1995) and Ringuest and Graves (2000) are also evaluated with the formulated model. Each set of randomly selected coefficient values represents one realized instance of the uncertain coefficients in the project portfolio selection problem. A total of 10,000 sets of randomly selected coefficient values are generated. Each solution is then evaluated by the constraints of the formulated model to determine if it is feasible. Only if the solution is feasible, it is evaluated with the three objective functions of the formulated model to obtain the corresponding values for each z_k. The criterion vectors $\mathbf{z}$ are then evaluated by (17) to calculate the corresponding value of $V_p(\mathbf{z})$ whereas z_k^{**} is also updated for the realized coefficient values. The percentage of feasible solutions over the 10,000 realizations of the uncertain coefficients, the average, worst, and the 10%, 50%, and 90% percentiles of $V_p(\mathbf{z})$ for the L_p-metric value functions with $p = 2$, $p = 4$ and $p = \infty$, are reported in Table 13, Table 14, and Table 15, respectively. Similar results on the individual objective functions are reported in Table E.1, Table E.2, and Table E.3 in Appendix E.

Insert Table 13, Table 14 & Table 15 Here

From these tables, it is first observed that non-robust solutions have generally low chance of feasibility. On the other hand, the robust solutions have generally very high percentage of feasibility. This is a critical issue because the feasibility of a solution is always the primary concern of any mathematical programming problem. In Table 13, Table 14, and Table 15, results for any budget of uncertainty higher than $\Gamma_i > 1.5$ are not reported because the feasibility of solutions is always satisfied when $\Gamma_i = 1.5$ and hence, further increase in Γ_i may deteriorate the quality of the solutions without the need to improve feasibility. In addition, for the L_2-metric value function in Table 13, 100% of the solutions are feasible when $\Gamma_i = 0.5$. Hence, the solutions obtained for $\Gamma_i > 0.5$ might not have been reported in this table. Note that every solution corresponding to $\Gamma_i > 0.5$, including solution (1,11,12,13,14) computed from $\Gamma_i = 1.5$, is always outperformed by solution (1,6,11,12,13,14) computed from $\Gamma_i = 0.5$.

Table 16 in the following presents the final robust solutions for the L_p-metric value functions with $p = 2$, $p = 4$ and $p = \infty$. In the table, the percentages of improvement in feasibility and the corresponding deteriorations in solution quality are presented. The relative deterioration is calculated with respect to the corresponding optimal nominal solutions.

Insert Table 16 Here

7. Conclusions

In this study, the problem of selecting a portfolio of R&D projects is considered when there are multiple conflicting objectives and when there are uncertainties in problem data including objective function and constraint coefficients. A robust augmented weighted Tchebycheff program is formulated and its linear counterpart is employed within the interactive robust weighted Tchebycheff procedure to generate robust nondominated solutions. The final portfolio is most preferred by the DM and is robust in terms of all possible realizations of uncertain problem coefficients. Through an illustrative example, the robust solutions are shown to have not only a very high percentage of feasibility but also a worst and average performance that is comparable to that of the nominal solutions.

An extraordinary strength of the proposed approach is that this robustness is achieved without bothering the DM in supplying unknown distribution details for the imprecise coefficients which is a major inconvenience in practical applications. Moreover, the proposed approach can be readily extended to other multiobjective mixed integer linear programming problems with uncertainties existing in both objective function and constraint coefficients. Therefore, a future research direction is to apply the proposed interactive robust weighted Tchebycheff procedure to such problems.

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Appendix A: Determining the robust ideal point

Proposition 2. The robust ideal point $\mathbf{z}^*$ can be determined from the following model for all $k = 1, \cdots, K$,

$$
\begin{aligned}
\min \quad & z_k & & \\
\text{s.t.} \quad & \sum_j \bar{c}_{kj} x_j + \Gamma'_k q'_k + \sum_j r'_{kj} \le z_k & \forall k & \\
& \sum_j \bar{a}_{ij} x_j + \Gamma_i q_i + \sum_j r_{ij} \le b_i & \forall i & \\
& q'_k + r'_{kj} \ge \hat{c}_{kj} y_j & \forall k, j & \\
& q_i + r_{ij} \ge \hat{a}_{ij} y_j & \forall i, j & \qquad \text{(A.1)} \\
& -y_j \le x_j \le y_j, \; x_j \in (0,1), \; y_j \ge 0 & \forall j & \\
& q_i \ge 0 & \forall i & \\
& q'_k \ge 0, \; z_k \text{ unrestricted} & \forall k & \\
& r'_{kj} \ge 0 & \forall k, j & \\
& r_{ij} \ge 0 & \forall i, j\,. &
\end{aligned}
$$

Proof. The robust ideal point $\mathbf{z}^*$ can be determined using the following non-linear model for all $k = 1, \cdots, K$,

$$
\begin{aligned}
\min \quad & z_k = \max_{\mathbf{c}_k} [\, \bar{f}_k(\mathbf{c}_k, \mathbf{x}) \mid \sum_j \delta'_{kj} \le \Gamma'_k] & & \\
\text{s.t.} \quad & \max_{\mathbf{a}_i} [\bar{g}_i(\mathbf{a}_i, \mathbf{x}) \mid \sum_j \delta_{ij} \le \Gamma_i] \le b_i & \forall i & \qquad \text{(A.2)} \\
& x_j \in \{0,1\} & \forall j\,. &
\end{aligned}
$$

Model (A.2) can be reformulated as model (A.3) in the following

$$
\begin{aligned}
\min \quad & z_k & & \\
\text{s.t.} \quad & \max_{\mathbf{c}_k} [\, \bar{f}_k(\mathbf{c}_k, \mathbf{x}) \mid \sum_j \delta'_{kj} \le \Gamma'_k] \le z_k & \forall k & \\
& \max_{\mathbf{a}_i} [\bar{g}_i(\mathbf{a}_i, \mathbf{x}) \mid \sum_j \delta_{ij} \le \Gamma_i] \le b_i & \forall i & \qquad \text{(A.3)} \\
& x_j \in \{0,1\} & \forall j & \\
& z_k \text{ unrestricted} & \forall k &
\end{aligned}
$$

Model (A.1) is directly obtained from model (A.3) by following the derivation of (8) from (7).■

Appendix B: Proof of Proposition 1

Model (12) is first reformulated as

$$
\begin{aligned}
\min \quad & d \\
\text{s.t.} \quad & d \geq \alpha + \rho \textstyle\sum_k \alpha_k && \forall k \\
& \alpha \geq w_k \alpha_k && \forall k \\
& \max_{\mathbf{c}_k} [\, \overline{f}_k(\mathbf{c}_k, \mathbf{x}) \,|\, \textstyle\sum_j \delta'_{kj} \leq \Gamma'_k \,] - \alpha_k \leq z_k^{**} && \forall k \\
& \max_{\mathbf{a}_i} [\, \overline{g}_i(\mathbf{a}_i, \mathbf{x}) \,|\, \textstyle\sum_j \delta_{ij} \leq \Gamma_i \,] \leq b_i && \forall i \\
& x_j \in \{0,1\} && \forall j\,.
\end{aligned}
\tag{B.1}
$$

Using the derivation of (8) from (7), (13) follows.■

Appendix C: Procedure for filtering a set of vectors

The following procedure reduces $2P$ robust criterion vectors to P most dispersed ones using the *d*-norm relative distance measure (Steuer 2003). In the procedure, Z^{2P} represents the set of the robust criterion vectors that have not been selected while Z^{P} the set of the most dispersed ones that have been selected.

Step 1. Calculate $\overline{\mathrm{z}}_k = \max\{\mathrm{z}_k \,|\, \mathbf{z} \in Z^{2P}\}$, $\underline{\mathrm{z}}_k = \min\{\mathrm{z}_k \,|\, \mathbf{z} \in Z^{2P}\}$, $\pi_k = (\overline{\mathrm{z}}_k - \underline{\mathrm{z}}_k)^{-1} / \sum_k (\overline{\mathrm{z}}_k - \underline{\mathrm{z}}_k)^{-1}$.

Randomly select a vector from Z^{2P} and transfer it to Z^{P}.

Step 2. Find $\mathbf{z}^{\max} \in \mathrm{Z}^{2P}$ so that $\mathbf{z}^{\max}$ is the most dissimilar vector from all vectors in Z^{P}, that is

$$\min_{\mathbf{z}' \in \mathrm{Z}^P} \left[\sum_{k=1}^{K} (\pi_k \,|\, \mathrm{z}_k^{\max} - \mathrm{z}'_k \,|)^d \right]^{1/d} = \max_{\mathbf{z} \in \mathrm{Z}^{2P}} \left[\min_{\mathbf{z}' \in \mathrm{Z}^P} \left[\sum_{k=1}^{K} (\pi_k \,|\, \mathrm{z}_k - \mathrm{z}'_k \,|)^d \right]^{1/d} \right]$$

Step 3. Transfer $\mathbf{z}^{\max}$ to Z^{P}. If $|\,\mathrm{Z}^{P}| = P$, then terminate; otherwise go to Step 2.

The procedure above is used in Step 3 of the interactive robust weighted Tchebycheff procedure to reduce $2P$ criterion vectors to P most dispersed ones. It is also used in Step 2 to reduce $20K$ weighting vectors to $2P$ most dispersed ones.

Appendix D: The basic formulation for the illustrative example

$$
\begin{aligned}
&\min\ -f_1(x) = -1600x_1 - 425x_2 - 213x_3 - 213x_4 - 2600x_5 - 750x_6 - 11x_7 - 11x_8 - 3x_9 - 18x_{10} \\
&\qquad\qquad - 40800x_{11} - 1200x_{12} - 3000x_{13} - 8000x_{14} - 85x_{3.4} - 3400x_{12.13.14} \\
&\min\ \ f_2(x) = 5x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6 + x_7 + x_8 + x_9 + x_{10} + 2x_{11} + x_{13} \\
&\min\ \ f_3(x) = 10200x_{11} + 300x_{12} + 750x_{13} + 2000x_{14} \\
&\text{s.t.} \\
&\quad 16000x_1 + 500x_2 + 350x_3 + 500x_4 + 2500x_5 + 1000x_6 - 268x_{3.4} - 350x_{4.5} \\
&\qquad\qquad - 250x_{4.6} - 250x_{5.6} - 600x_{4.5.6} \le 20000 \\
&\quad 3250x_1 + 1000x_2 + 350x_3 + 500x_4 + 2500x_5 + 1000x_6 + 28x_7 + 27x_8 + 7x_9 + 44x_{10} \\
&\qquad\qquad -155x_{2.3} - 225x_{2.4} - 188x_{3.4} - 200x_{4.5} - 175x_{4.6} - 125x_{5.6} - 375x_{4.5.6} \le 6000 \\
&\quad -x_1 + x_2 \le 0,\ -x_2 + x_3 \le 0,\ x_3 - x_4 \le 0,\ -x_5 + x_7 \le 0,\ -x_5 + x_8 \le 0,\ -x_5 + x_9 \le 0, \\
&\quad -x_5 + x_{10} \le 0,\ -x_1 + x_{11} \le 0,\ -x_{11} + x_{12} \le 0,\ -x_{11} + x_{13} \le 0,\ -x_{11} + x_{14} \le 0, \\
&\quad x_2 + x_3 - x_{2.3} \le 1,\ -x_2 - x_3 + 2x_{2.3} \le 0,\ x_2 + x_4 - x_{2.4} \le 1,\ -x_2 - x_4 + 2x_{2.4} \le 0, \\
&\quad x_3 + x_4 - x_{3.4} \le 1,\ -x_3 - x_4 + 2x_{3.4} \le 0,\ x_4 + x_5 - (x_{4.5} + x_{4.5.6}) \le 1,\ -x_4 - x_5 + 2(x_{4.5} + x_{4.5.6}) \le 0, \\
&\quad x_4 + x_6 - (x_{4.6} + x_{4.5.6}) \le 1,\ -x_4 - x_6 + 2(x_{4.6} + x_{4.5.6}) \le 0,\ x_5 + x_6 - (x_{5.6} + x_{4.5.6}) \le 1, \\
&\quad -x_5 - x_6 + 2(x_{5.6} + x_{4.5.6}) \le 0,\ x_4 + x_5 + x_6 - x_{4.5.6} \le 2,\ -x_4 - x_5 - x_6 + 3x_{4.5.6} \le 0, \\
&\quad x_{1.2} + x_{1.3} + x_{1.4} - x_{12.13.14} \le 2,\ -x_{12} - x_{13} - x_{14} + 3x_{12.13.14} \le 0, \\
&\quad x_1 = 1 \\
&\quad x_j = (0,1) \quad \forall j
\end{aligned}
\tag{D.1}
$$

Appendix E: Robustness in objective functions of selected solutions

Table E.1, Table E.2, and Table E.3 in this appendix report the results of the simulation study on the values of z_k for $k = 1,2,3$, *i.e.*, the values of the three individual objective functions in the illustrative example. Results in these tables are reported in a similar manner as those for the L_p-metric value functions. However, in addition to the percentage of feasible solutions, the average, worst, and the 10%, 50%, and 90% percentiles of the values of each z_k, the nominal and robust values of each z_k are also reported. Each of the three components of the robust criterion vectors $\overline{\mathbf{z}} \in Z^{\Gamma,\Gamma'}$ is reported in a separate table. Results in these tables confirm that the performance of the robust criterion vectors is comparable to that of the nominal solutions.

Insert Table E.1, Table E.2 & Table E.3 Here

Table 1. Original estimates for independent benefits, costs, and risk scores

Project	Mandated	Contingent upon	Annual Benefits	Hardware costs	Software costs	Miscellaneous costs	Risk scores
1	Yes	-	1600	16,000	3250	0	5
2	No	1	425	500	1000	0	4
3	No	2	213	350	350	0	3
4	No	2	213	500	500	0	3
5	No	-	2600	2500	2500	0	3
6	No	-	750	1000	1000	0	3
7	No	5	11	0	28	0	1
8	No	5	11	0	27	0	1
9	No	5	3	0	7	0	1
10	No	5	18	0	44	0	1
11	No	1	40,800	0	0	10,200	2
12	No	11	1200	0	0	300	0
13	No	11	3000	0	0	750	1
14	No	11	8000	0	0	2000	0

Table 2. Original estimates for interdependent costs and benefits

Interdependent projects	Additional benefits	Shared Hardware costs	Shared software costs
2, 3			155
2, 4			225
3, 4	85	268	188
4, 5		350	200
4, 6		250	175
5, 6		250	125
12, 13, 14	3400		
4, 5, 6		600†	375†

† Original values changed to match the formulation of Appendix D.

Table 3. Comparison among solutions found by the three methods

No	Benefit	Risk	Cost	Projects selected	S&K	R&G	H&N&S	Status
1	1600	5	0	1			✓	Nondominated
2	4200	8	0	1,5	✓		✓	Nondominated
3	4218	9	0	1,5,10		✓	✓	Nondominated
4	4229	10	0	1,5,7,10		✓	✓	Nondominated
5	4240	11	0	1,5,7,8,10			✓	Nondominated
6	4243	12	0	1,5,7,8,9,10			✓	Nondominated
7	42400	7	10200	1,11	✓		✓	Nondominated
8	43600	7	10500	1,11,12		✓	✓	Nondominated
9	45000	10	10200	1,5,11			✓	Nondominated
10	45018	11	10200	1,5,10,11			✓	Nondominated
11	45040	13	10200	1,5,7,8,10,11		✓	✓	Nondominated
12	45043	14	10200	1,5,7,8,9,10,11			✓	Nondominated
13	46200	10	10500	1,5,11,12			✓	Nondominated
14	46218	11	10500	1,5,10,11,12			✓	Nondominated
15	46240	13	10500	1,5,7,8,10,11,12		✓	✓	Nondominated
16	46243	14	10500	1,5,7,8,9,10,11,12			✓	Nondominated
17	46600	8	11250	1,11,12,13			✓	Nondominated
18	48000	11	10950	1,5,11,13	✓			Nondominated
19	48040	14	10950	1,5,7,8,10,11,13		✓	✓	Nondominated
20	48043	15	10950	1,5,7,8,9,10,11,13			✓	Nondominated

21	49200	11	11250	1,5,11,12,13			✓	Nondominated
22	49240	14	11250	1,5,7,8,10,11,12,13			✓	Nondominated
23	49243	15	11250	1,5,7,8,9,10,11,12,13			✓	Nondominated
24	50400	7	12200	1,11,14	✓		✓	Nondominated
25	51600	7	12500	1,11,12,14	✓		✓	Nondominated
26	53000	10	12200	1,5,11,14	✓		✓	Nondominated
27	53018	11	12200	1,5,10,11,14		✓	✓	Nondominated
28	53029	12	12200	1,5,8,10,11,14		✓		Nondominated
29	53040	13	12200	1,5,7,8,10,11,14			✓	Nondominated
30	53043	14	12200	1,5,7,8,9,10,11,14			✓	Nondominated
31	53400	8	12950	1,11,13,14			✓	Nondominated
32	54200	10	12500	1,5,11,12,14			✓	Nondominated
33	54240	13	12500	1,5,7,8,10,11,12,14		✓	✓	Nondominated
34	54243	14	12500	1,5,7,8,9,10,11,12,14			✓	Nondominated
35	56000	11	12950	1,5,11,13,14		✓	✓	Nondominated
36	56018	12	12950	1,5,10,11,13,14		✓		Nondominated
37	56029	13	12950	1,5,7 *or* 8,10,11,13,14		✓		Nondominated
38	56040	14	12950	1,5,7,8,10,11,13,14			✓	Nondominated
39	56043	15	12950	1,5,7,8,9,10,11,13,14			✓	Nondominated
40	58000	8	13250	1,11,12,13,14	✓		✓	Nondominated
41	60600	11	13250	1,5,11,12,13,14			✓	Nondominated
42	60618	12	13250	1,5,10,11,12,13,14		✓		Nondominated
43	60629	13	13250	1,5,7,10,11,12,13,14		✓		Nondominated
44	60640	14	13250	1,5,7,8,10,11,12,13,14			✓	Nondominated
45	60643	15	13250	1,5,7,8,9,10,11,12,13,14	✓	✓	✓	Nondominated
46	4232	11	0	1,5,8,9,10		✓		Dominated by No 5
47	43150	10	10200	1,6,11		✓		Dominated by No 9
48	44350	10	10500	1,6,11,12		✓		Dominated by No 13
49	45022	12	10200	1,5,7,8,11		✓		Dominated by 1,5,8,10,11 (CE)
50	46214	12	10500	1,5,8,9,11,12		✓		Dominated by No 14
51	48011	12	10950	1,5,8,11,13		✓		Dominated by 1,5,10,11,13 (CE)
52	48022	13	10950	1,5,7,8,11,13		✓		Dominated by 1,5,8,10,11,13 (CE)
53	48032	14	10950	1,5,7,9,10,11,13	✓			Dominated by No 19
54	49211	12	11250	1,5,7,11,12,13		✓		Dominated by 1,5,10,11,12,13 (CE)
55	49222	13	11250	1,5,7,8,11,12,13		✓		Dominated by 1,5,8,10,11,12,13 (CE)
56	49232	14	11250	1,5,7,9,10,11,12,13		✓		Dominated by No 22
57	52350	10	12500	1,6,11,12,14		✓		Dominated by No 26
58	53032	13	12200	1,5,7,9,10,11,14		✓		Dominated by No 29
59	54203	11	12500	1,5,9,11,12,14		✓		Dominated by 1,5,10,11,12,14 (CE)
60	54222	12	12500	1,5,7,8,11,12,14		✓		Dominated by 1,5,8,10,11,12,14 (CE)
61	56032	14	12950	1,5,7 *or* 8,9,10,11,13,14		✓		Dominated by No 38
62	60632	14	13250	1,5,8,9,10,11,12,13,14		✓		Dominated by No 44

Table 4. Performance summary of the three methods

Method	Feasible solutions	Nondominated solutions	Nondominated criterion vectors
Santhanam and Kyparisis (S&K)	9	8	8
Ringuest and Graves (R&G)	100	16	15
Augmented weighted Tchebycheff (H&N&S)	39	39	39
Complete enumeration (CE)	234	63	54

Table 5. Deterministic solutions for the L_2-metric value function

Source	Selected solution	Benefit	Risk	Cost	$V_2(\mathbf{z})$
H&N&S Iter. 1-2	1,5,11,12,14	54,200	10	12,500	15,781.1605
H&N&S Iter. 3	1,11,12,13,14	58,000	8	13,250	15,946.6906
H&N&S Iter. 4-8	1,5,7,8,10,11,12,13,14	60,640	14	13,250	16,024.9983
R&G	1,5,7,8,9,10,11,12,13,14	60,643	15	13,250	16,024.9980
S&K	1,5,7,8,9,10,11,12,13,14	60,643	15	13,250	16,024.9980
CE	1,5,7,8,10,11,12,13,14	60,640	14	13,250	16,024.9983

Table 6. Deterministic solutions for the L_4-metric value function

Source	Selected solution	Benefit	Risk	Cost	$V_4(\mathbf{z})$
H&N&S Iter. 1	1,11,12,14	51,600	7	12,500	16,016.0299
H&N&S Iter. 2	1,5,11,12,14	54,200	10	12,500	16,185.5093
H&N&S Iter. 3-8	1,5,7,8,9,10,11,14	53,043	14	12,200	16,209.3660
R&G	1,5,7,9,10,11,14	53,032	13	12,200	16,208.6466
S&K	1,5,11,14	53,000	10	12,200	16,206.5382
CE	1,5,7,8,9,10,11,14	53,043	14	12,200	16,209.3660

Table 7. Deterministic solutions for the L_∞-metric value function

Source	Selected solution	Benefit	Risk	Cost	$V_\infty(\mathbf{z})$
H&N&S Iter. 1	1,11,12,14	51,600	7	12,500	16,250.0000
H&N&S Iter. 2	1,5,11,12,14	54,200	10	12,500	16,250.0000
H&N&S Iter. 3	1,11,14	50,400	7	12,200	16,340.0000
H&N&S Iter. 4-8	1,5,7,8,9,10,11,12,13	49,243	15	11,250	16,580.0000
R&G	1,5,7,9,10,11,12,13	49,232	14	11,250	16,576.7000
S&K	1,11,14	50,400	7	12,200	16,340.0000
CE	1,5,7,8,9,10,11,12,13	49,243	15	11,250	16,580.0000

Table 8. Half-interval widths for independent benefits, costs, and risk scores

Project	Annual Benefits	Hardware costs	Software costs	Miscellaneous costs	Risk scores
1	320	2575	710	0	1.25
2	42.5	150	350	0	0.6
3	10.65	105	122.5	0	0.3
4	10.65	150	175	0	0.3
5	260	750	875	0	0.3
6	75	300	350	0	0.3
7	1.1	0	9.8	0	0.05
8	1.1	0	9.45	0	0.05
9	0.09	0	2.45	0	0.1
10	1.8	0	15.4	0	0.1
11	10200	0	0	3060	0.2
12	120	0	0	30	0
13	300	0	0	75	0.1
14	1600	0	0	300	0

Table 9. Half-interval widths for interdependent costs and benefits

Interdependent projects	Additional benefits	Shared Hardware costs	Shared software costs
2, 3			54.25
2, 4			78.75
3, 4	4.25	80.4	65.8
4, 5		105	70
4, 6		75	61.25
5, 6		75	43.75
12, 13, 14	1020		
4, 5, 6		75	43.75

Table 10. Robust solutions for the L_2-metric value function

Γ_i	Selected solution	Nominal benefit	Nominal risk	Nominal cost	$V_2(\mathbf{z})$
0.5	1,6,11,12,13,14	58,750	11	13,250	15,984.6369
1.0	1,6,11,12,13,14	58,750	11	13,250	15,984.6369
1.5	1,11,12,13,14	58,000	8	13,250	15,946.6906

Table 11. Robust solutions for the L_4-metric value function

Γ_i	Selected solution	Nominal benefit	Nominal risk	Nominal cost	$V_4(\mathbf{z})$
0.5	1,2,3,6,11,14	51,788	17	12,200	16,108.8844
1.0	1,6,11,12,14	52,350	10	12,500	16,080.2516
1.5	1,6,11,12,14	52,350	10	12,500	16,080.2516

Table 12. Robust solutions for L_∞-metric value function

Γ_i	Selected solution	Nominal benefit	Nominal risk	Nominal cost	$V_\infty(\mathbf{z})$
0.5	1,2,3,4,11,14	51,336	17	12,200	16,340.0000
1.0	1,11,14	50,400	7	12,200	16,340.0000
1.5	1,11,14	50,400	7	12,200	16,340.0000

Table 13. Solution performance for L_2-metric value function

Solution	Source	Feasibility Percentage	Percentiles 10%	Percentiles 50%	Percentiles 90%	Average	Worst
1,5,7,8,9,10,11,12,13,14	R&G/S&K	46.99%	15,303.0	16,021.3	16,764.5	16,029.6	15,026.5
1,5,7,8,10,11,12,13,14	Nominal	47.20%	15,303.5	16,020.4	16,763.8	16,029.2	15,026.5
1,6,11,12,13,14	Robust (Γ_i=0.5,1.0)	100.00%	15,260.7	15,984.8	16,710.3	15,984.3	14,972.8
1,11,12,13,14	Robust (Γ_i=1.5)	100.00%	15,227.5	15,947.2	16,662.9	15,945.7	14,937.6

Table 14. Solution performance for L_4-metric value function

Solution	Source	Feasibility Percentage	Percentiles 10%	Percentiles 50%	Percentiles 90%	Average	Worst
1,5,7,8,9,10,11,14	Nominal	46.99%	15,535.9	16,200.9	16,824.0	16,194.3	15,259.8
1,5,7,9,10,11,14	R&G	48.02%	15,534.5	16,200.0	16,823.2	16,192.8	15,259.4
1,5,11,14	S&K	51.65%	15,532.7	16,199.1	16,819.8	16,191.6	15,257.9
1,2,3,6,11,14	Robust (Γ_i =0.5)	78.26%	15,460.0	16,094.1	16,654.7	16,078.8	15,181.0
1,6,11,12,14	Robust (Γ_i=1.0,1.5)	100.00%	15,410.8	16,071.6	16,672.0	16,058.6	15,129.3

Table 15. Solution performance for L_∞ -metric value function

Solution	Source	Feasibility Percentage	Percentiles 10%	50%	90%	Average	Worst
1,5,7,8,9,10,11,12,13	Nominal	46.99%	15,890.2	16,330.3	16,821.2	16,343.5	15,687.1
1,5,7,9,10,11,12,13	R&G	48.02%	15,887.7	16,327.6	16,818.8	16,341.2	15,687.1
1,2,3,4,11,14	Robust (Γ_i =0.5)	99.99%	15,608.8	16,339.4	17,026.6	16,332.7	15,338.2
1,11,14	S&K/Robust (Γ_i =1.0,1.5)	100.00%	15,608.8	16,339.4	16,908.7	16,299.9	15,338.2

Table 16. Preferred robust solutions for L_p -metric value functions

p	Robust solution	Improvement in feasibility percentage	Deterioration in value function: Average	Deterioration in value function: Worst
2	1,6,11,12,13,14	52.80%	0.28%	0.36%
4	1,2,3,6,11,14	31.27%	0.71%	0.52%
4	1,6,11,12,14	53.01%	0.84%	0.86%
∞	1,2,3,4,11,14	53.00%	0.07%	2.22%
∞	1,11,14	53.01%	0.27%	2.22%

Table E.1. Benefit function performance for deterministic and robust solutions

p	Solution	Source	Feasibility Percentage	Nominal Benefit	Robust Benefit	Percentiles 10%	50%	90%	Average	Worst
2	1,5,7,8,10,11,12,13,14	Nominal	47.20%	60,640	-	52,400.4	60,504.7	68,846.8	60,622.3	47,817.5
2	1,5,7,8,9,10,11,12,13,14	R&G/S&K	46.99%	60,643	-	52,403.4	60,507.8	68,849.9	60,625.1	47,820.4
2	1,6,11,12,13,14	Robust (Γ_i =0.5,1.0)	100.00%	58,750	51,610	50,563.8	58,756.2	66,899.9	58,760.0	46,073.9
2	1,11,12,13,14	Robust (Γ_i =1.5)	100.00%	58,000	50,860	49,799.7	58,003.2	66,149.0	58,010.4	45,359.9
4	1,5,7,8,9,10,11,14	Nominal	46.99%	53,043	-	44,868.4	52,918.0	61,246.8	53,023.0	40,976.6
4	1,5,7,9,10,11,14	R&G	48.02%	53,032	-	44,861.8	52,926.9	61,245.1	53,023.2	40,965.8
4	1,5,11,14	S&K	51.65%	53,000	-	44,806.4	52,943.7	61,166.3	53,003.6	40,934.7
4	1,2,3,6,11,14	Robust (Γ_i =0.5)	78.26%	51,788	44,648	43,591.1	51,762.3	59,947.2	51,799.3	39,892.1
4	1,6,11,12,14	Robust (Γ_i =1.0,1.5)	100.00%	52,350	45,210	44,129.7	52,355.2	60,525.6	52,353.0	40,535.3
∞	1,5,7,8,9,10,11,12,13	Nominal	46.99%	49,243	-	41,112.1	49,123.0	57,374.7	49,220.1	38,473.4
∞	1,5,7,9,10,11,12,13	R&G	48.02%	49,232	-	41,103.2	49,115.3	57,376.3	49,220.0	38,463.1
∞	1,2,3,4,11,14	Robust (Γ_i =0.5)	99.99%	51,336	44,196	43,112.6	51,334.5	59,473.8	51,340.0	39,455.2
∞	1,11,14	S&K/Robust (Γ_i =1.0,1.5)	100.00%	50,400	43,260	42,175.5	50,393.8	58,538.6	50,403.2	38,494.5

Table E.2. Risk function performance for deterministic and robust solutions

p	Solution	Source	Feasibility Percentage	Nominal Risk	Robust Risk	Percentiles 10%	50%	90%	Average	Worst
2	1,5,7,8,10,11,12,13,14	Nominal	47.20%	14.000	-	12.969	13.995	15.018	13.993	15.755
2	1,5,7,8,9,10,11,12,13,14	R&G/S&K	46.99%	15.000	-	13.958	14.989	16.033	14.993	16.843
2	1,6,11,12,13,14	Robust (Γ_i =0.5,1.0)	100.00%	11.000	11.875	9.972	10.987	11.997	10.985	12.737
2	1,11,12,13,14	Robust (Γ_i =1.5)	100.00%	8.000	8.875	6.983	7.991	8.989	7.987	9.484
4	1,5,7,8,9,10,11,14	Nominal	46.99%	14.000	-	12.962	13.993	15.021	13.994	15.792
4	1,5,7,9,10,11,14	R&G	48.02%	13.000	-	11.973	12.993	14.023	12.995	14.831
4	1,5,11,14	S&K	51.65%	10.000	-	8.983	9.994	11.009	9.992	11.677
4	1,2,3,6,11,14	Robust (Γ_i =0.5)	78.26%	17.000	17.875	15.848	16.982	18.107	16.986	19.375
4	1,6,11,12,14	Robust (Γ_i =1.0,1.5)	100.00%	10.000	10.875	8.977	9.986	10.997	9.986	11.705
∞	1,5,7,8,9,10,11,12,13	Nominal	46.99%	15.000	-	13.958	14.989	16.033	14.993	16.843
∞	1,5,7,9,10,11,12,13	R&G	48.02%	14.000	-	12.960	13.993	15.028	13.995	15.882
∞	1,2,3,4,11,14	Robust (Γ_i =0.5)	99.99%	17.000	17.875	15.852	16.976	18.108	16.982	19.382
∞	1,11,14	S&K/Robust (Γ_i =1.0,1.5)	100.00%	7.000	7.875	5.981	6.995	7.989	6.988	8.439

Table E.3. Cost function performance for deterministic and robust solutions

p	Solution	Source	Feasibility Percentage	Nominal Cost	Robust Cost	Percentiles 10%	50%	90%	Average	Worst
2	1,5,7,8,10,11,12,13,14	Nominal	47.20%	13,250	-	10,785.5	13,264.7	15,653.3	13,236.2	16,578.3
2	1,5,7,8,9,10,11,12,13,14	R&G/S&K	46.99%	13,250	-	10,784.9	13,262.3	15,656.6	13,234.5	16,578.3
2	1,6,11,12,13,14	Robust (Γ_i =0.5,1.0)	100.00%	13,250	15,392	10,799.1	13,251.9	15,684.5	13,247.8	16,636.2
2	1,11,12,13,14	Robust (Γ_i =1.5)	100.00%	13,250	15,392	10,799.1	13,251.9	15,684.5	13,247.8	16,636.2
4	1,5,7,8,9,10,11,14	Nominal	46.99%	12,200	-	9,732.6	12,205.5	14,602.8	12,185.0	15,531.5
4	1,5,7,9,10,11,14	R&G	48.02%	12,200	-	9,729.2	12,205.5	14,609.6	12,187.2	15,531.5
4	1,5,11,14	S&K	51.65%	12,200	-	9,731.0	12,197.8	14,611.9	12,182.2	15,531.5
4	1,2,3,6,11,14	Robust (Γ_i =0.5)	78.26%	12,200	14,342	9,765.0	12,213.6	14,649.9	12,209.2	15,539.2
4	1,6,11,12,14	Robust (Γ_i =1.0,1.5)	100.00%	12,500	14,642	10,045.6	12,507.6	14,931.3	12,497.9	15,825.7
∞	1,5,7,8,9,10,11,12,13	Nominal	46.99%	11,250	-	8,791.8	11,243.0	13,652.8	11,233.9	14,376.4
∞	1,5,7,9,10,11,12,13	R&G	48.02%	11,250	-	8,786.9	11,244.4	13,653.9	11,235.9	14,376.4
∞	1,2,3,4,11,14	Robust (Γ_i =0.5)	99.99%	12,200	14,342	9,750.6	12,202.0	14,637.2	12,198.2	15,539.2
∞	1,11,14	S&K/Robust (Γ_i =1.0,1.5)	100.00%	12,200	14,342	9,750.6	12,202.0	14,636.3	12,198.1	15,539.2