Development of a load sharing policy by managing the residual life based on a stochastic process

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In this paper, we analyze the time (*viz.*, the number of cycles) to reach any given crack size in a fatigue life test using a gamma stochastic process. It is assumed that the time increments are non-stationary but independent for each specimen while the shape parameter of the gamma distribution is a function of the crack length. In addition, using a random effect model, the between-specimen variability is explained by modeling the scale parameter of the process with a gamma distribution. This yields explicit formulas for the marginal lifetime distributions, the associated mean and variance, which boosts computational efficiency.

Keywords: fatigue crack growth, gamma distribution, lifetime estimation, Paris law, reliability, stochastic process

JEL Classifications: C13, C16, C24

1 Introduction

In the industrial and manufacturing design, modeling the fatigue crack growth properties has been an important research problem since failure risk and reliability analyses, inspection planning and maintenance policy all depend on the accuracy of the probabilistic model describing the material behavior. Different modeling approaches have been studied by numerous researchers [1–24]. In particular, Kozin and Bogdanoff [11] stated that (i) for a single specimen, the crack growth evolution over time (*viz.*, the number of cycles) can be modeled as a stochastic Markov process; (ii) the between-specimen variability of crack growth behavior can be modeled by treating suitable parameters of the stochastic process model as random quantities; and (iii) modeling the time to first reach a given crack length as a stochastic process over the crack length domain is the proper approach to study fatigue crack growth.

Here, a non-homogeneous gamma process model is studied for analyzing the time to reach a given crack length in fatigue life test under constant amplitude cycling loading. That is, time increments are non-stationary independent from gamma distributions where the shape parameter is an appropriate function of the crack length and the scale parameter is appropriately chosen such that the process mean gives exactly the deterministic crack growth model of one's choice (*e.g.*, Paris regime). In addition, using a random effect model, the between-specimen variability is explained by modeling the scale parameter of the process with a gamma distribution. This yields closed-form solutions for the marginal lifetime distributions, the associated moments of the stochastic process, which boosts computational efficiency.

The correlation often observed between the estimates of the Paris law parameters C and m is also discussed. Here, a scheme to decorrelate the estimates of Paris law parameters is examined as it is necessary for the random effect model to account for the between-specimen variability by randomizing the crack growth rate parameter. Using this scheme, the Paris law is scalized so that the estimation method generates quasi-uncorrelated estimates of C and m. All the model parameters were estimated under the maximum likelihood principle, using the experimental datasets produced by Virkler *et al.* [21] for model validation. Model assumptions and goodness-of-fit were assessed, and no violation was found in general, suggesting that the proposed model is effective in analyzing and interpreting the stochastic behavior of the crack growth evolution.

2 Model Descriptions and MLE

Here, the first passage time $T(a)$ for a specimen to reach a crack length a is assumed to be governed by a non-homogeneous gamma process with constant scale parameter and shape parameter which depends on a. A random variable T has a gamma distribution with shape parameter $\eta > 0$ and scale parameter $\varphi > 0$ if its probability density function is given by

$$f(t;\ \eta, \varphi) = \frac{1}{\Gamma(\eta)\varphi^{\eta}} t^{\eta-1} e^{-t/\varphi}, \qquad t > 0.$$

Let $\eta(a)$ be a non-decreasing, right-continuous, real-valued function for $a \geq a_0$ with $\eta(a_0) = 0$. The gamma process with shape function $\eta(a)$ and scale parameter φ is a continuous-time stochastic process $\{T(a);\ a \geq a_0\}$ with the following properties: (i) $T(a_0) = 0$; (ii) $T(a_2) - T(a_1) \sim Gamma(\eta(a_2) - \eta(a_1), \varphi)$ for all $a_2 > a_1 \geq a_0$; (iii) $T(a)$ has independent increments. The mean and variance of $T(a)$ are $E[T(a)] = \varphi\ \eta(a)$ and $Var[T(a)] = \varphi^2\ \eta(a)$, respectively. Hence, the variance-to-mean ratio is φ, which does not depend on a. Now, in order to formulate $\eta(a)$ explicitly, we start the derivation from the Paris law, which is given as

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = C'\ [\Delta K(a)]^{m'} \tag{1}$$

where C' and m' are model parameters, $\Delta K(a) = \Delta P\, Y(a)$ is the stress-intensity factor range with ΔP the load range, and $Y(a)$ a function which takes into account the shape of the crack and of the specimen. In the present work, a *scaled* form of the Paris law is defined, which is given as

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = C\left[\frac{\Delta K(a)}{\Delta K_0}\right]^m \tag{2}$$

where the scaling factor ΔK_0 is uniquely determined by imposing that the estimators of C and m are (asymptotically) uncorrelated. Under the stochastic framework, the Paris law assumption amounts to $E[\mathrm{d}A(t)/\mathrm{d}t|A(t) = a(t)] = C\ [\Delta K(a)/\Delta K_0]^m$. Hence, for the gamma process proposed here, the shape parameter $\eta(a)$ is given by

$$\eta(a) = \xi^{-1}\int_{a_0}^{a}\left[\frac{\Delta K(u)}{\Delta K_0}\right]^{-m}\mathrm{d}u \tag{3}$$

and scale parameter $\varphi = \xi C^{-1}$. Then, the first-order probability density function of $T(a)$ is given by

$$f_T(t) = \frac{1}{\Gamma(\eta(a))(\xi C^{-1})^{\eta(a)}}t^{\eta(a)-1}\exp\left[-\frac{t}{\xi C^{-1}}\right], \qquad t > 0,\ a > a_0 \tag{4}$$

with the mean and variance of $T(a)$ given by

$$E[T(a)] = \xi C^{-1}\eta(a) = C^{-1}\int_{a_0}^{a}\left[\frac{\Delta K(u)}{\Delta K_0}\right]^{-m}\mathrm{d}u \tag{5}$$

$$Var[T(a)] = \xi^2 C^{-2}\eta(a) = \xi C^{-2}\int_{a_0}^{a}\left[\frac{\Delta K(u)}{\Delta K_0}\right]^{-m}\mathrm{d}u \tag{6}$$

Hence, $E[T(a)]$ matches with the deterministic (scaled) Paris law for the time to reach a given crack length a.

The maximum likelihood estimation method is used to estimate the model parameters. Let t_{a_j} denote the observed time for an individual specimen to reach the crack length a_j $(j = 1,\ldots,M)$. From the properties (ii) and (iii) of gamma process, the increments $\Delta t_j = t_{a_j} - t_{a_{j-1}}$ $(j = 1,\ldots,M)$ $(t_{a_0} = 0)$ are independent gamma variables with shape parameters $\Delta\eta_j = \eta(a_j) - \eta(a_{j-1})$ and common scale $\varphi = \xi C^{-1}$. Then, the likelihood function of the observed data is given by

$$L(C, m, \xi|\text{data}) = \left[\prod_{j=1}^{M}\frac{(\Delta t_j)^{\Delta\eta_j - 1}}{\Gamma(\Delta\eta_j)}\right](\xi C^{-1})^{-\eta(a_M)}\exp\left(-\frac{t_{a_M}}{\xi C^{-1}}\right) \tag{7}$$

The MLE $\widehat{C}$, $\widehat{m}$ and $\widehat{\xi}$ can be obtained by maximizing the logarithm of (7), where both $\Delta\eta_j = \xi^{-1}\int_{a_{j-1}}^{a_j}[\Delta K(u)/\Delta K_0]^{-m}\mathrm{d}u$ and $\eta(a_M) = \xi^{-1}\int_{a_0}^{a_M}[\Delta K(u)/\Delta K_0]^{-m}\mathrm{d}u$ involve the parameters m and ξ.

2.1 Random effect for between-specimen variability

Although the within-specimen variability is explained by the proposed gamma process, a between-specimen variability cannot be explained by the (fixed effect) model. Here, it is assumed that the parameter C varies randomly across specimens following a gamma distribution with parameters δ and γ^{-1} so that the marginal density of T_a is obtained in closed-form as well as its moments. That is, the density function of C is given as

$$f(C) = \frac{\gamma^\delta}{\Gamma(\delta)} C^{\delta-1} e^{-\gamma C}, \qquad C > 0 \tag{8}$$

so that the density function of T_a is explicitly obtained as

$$f(t_a) = \frac{(\gamma\xi)^\delta}{B(\eta(a),\delta)} \frac{t_a^{\eta(a)-1}}{[t_a + \gamma\xi]^{\eta(a)+\delta}}, \qquad a > a_0 \tag{9}$$

The mean and variance of T_a are then

$$E[T(a)] = \gamma\xi\eta(a)/(\delta-1) \quad \text{for } \delta > 1 \tag{10}$$

$$Var[T(a)] = (\gamma\xi)^2\eta(a)[\eta(a)+\delta-1]/[(\delta-1)^2(\delta-2)] \quad \text{for } \delta > 2 \tag{11}$$

It is also known that the random variable $W = \delta\ T_a/[\gamma\ \xi\ \eta(a)]$ has an F distribution with $\nu_1 = 2\eta(a)$ and $\nu_2 = 2\delta$ degrees of freedom. Therefore, the cumulative distribution function of T_a can be written as

$$P(T_a \le t_a) = P\left(W \le \frac{\delta\ t_a}{\gamma\ \xi\ \eta(a)}\right) = F_{2\eta(a);2\delta}\left[\frac{\delta\ t_a}{\gamma\ \xi\ \eta(a)}\right] \tag{12}$$

For any fixed time t, A_t the crack length at time t is a random variable over the population of specimens, and it follows that

$$P(A_t > a) = P(T_a \le t) = F_{2\eta(a);2\delta}\left[\frac{\delta\ t}{\gamma\ \xi\ \eta(a)}\right] \tag{13}$$

which is known as *crack exceedance probability.*

Again, the maximum likelihood estimation method is used to estimate the model parameters including the hyper-parameters δ and γ for the distribution of C. Let $t_{a_{ij}}$ denote the observed times to reach the crack lengths $a_{i1} < \cdots < a_{iM_i}$ for the i-th specimen $(i = 1,\dots,N)$. Conditioned on the random effect C, the increments $\Delta t_{ij} = t_{a_{ij}} - t_{a_{ij-1}}$, $j = 1,\dots,M_i$ for the specimen i are independent random variables having a gamma distribution $Gamma(\Delta t_{ij};\ \Delta\eta_{ij}, \xi C^{-1})$, so that the likelihood function is given by (7). Averaging this likelihood over the distribution of the random parameter C, which is assumed to be $Gamma(C;\ \gamma^{-1},\delta)$, it is obtained

$$L(\xi,m,\gamma,\delta|\text{data}_i) = \frac{(\xi\gamma)^\delta\Gamma(\delta+\eta_i(a_{M_i}))}{\Gamma(\delta)\prod_{j=1}^{M_i}\Gamma(\Delta\eta_{ij})} \frac{\prod_{j=1}^{M_i}(\Delta t_{ij})^{\Delta\eta_{ij}-1}}{(\xi\gamma + t_{iM_i})^{\delta+\eta_i(a_{M_i})}} \tag{14}$$

Then, the likelihood function for all the N observed sample paths is obtained to be

$$L(\xi, m, \gamma, \delta|\text{data}) = \prod_{i=1}^{N} L_i(\xi, m, \gamma, \delta|\text{data}_i). \tag{15}$$

The MLE $\widehat{\xi}$, $\widehat{m}$, $\widehat{\gamma}$ and $\widehat{\delta}$ are obtained by maximizing the logarithm of (15).

2.2 Decorrelated MLE of C and m

Decorrelating the estimates of C and m is discussed in this section since it is necessary for the random effect model to account for the between-specimen variability by randomizing the crack growth rate parameter C across a population of specimens. Here, the scale factor ΔK_0 in (2) is determined such that the MLE of C and m are quasi-uncorrelated across a population of N specimens. In this way, decorrelated and conventional estimates share common statistical properties in repeated sampling. To determine ΔK_0, let us first consider the case when the method of least squares is applied for estimating the Paris law parameters after linearization of the randomized, scaled Paris equation:

$$\log(\mathrm{d}a/\mathrm{d}t)_{a_j} = \log C + m(\log \Delta K(a_j) - \log \Delta K_s) + \log X(t) \tag{16}$$

where $X(t)$ is usually assumed to be a stationary lognormal stochastic process. It is known that the least squares method generates correlated estimates of $\log \tilde{C}$ and $\tilde{m}$. This correlation can be completely eliminated by assuming $\log \Delta K_s \equiv \log \Delta K_0 = M^{-1} \sum_{j=1}^{M} \log \Delta K(a_j)$. Thus, in such a framework, the decorrelating factor $\Delta K_0 = \exp\left(M^{-1} \sum_{j=1}^{M} \log \Delta K(a_j)\right)$ can be computed before estimating C and m. Also, ΔK_0 is inside the range $(\Delta K(a_{\min}), \Delta K(a_{\max}))$. In addition, since $\Delta K(a_j) = \Delta P\, Y(a_j)$, it is easy to see that ΔK_0 is the product of ΔP by the geometric mean of the values $Y(a_j)$ $(j = 1, \ldots, M)$ so that the ratio $\Delta K/\Delta K_0$ is independent of load conditions. When multiple specimens are considered, if the crack lengths a_j $(j = 1, \ldots, M)$ at which the empirical rate is estimated are the same across all the paths, individually decorrelated estimates $(\log \tilde{C}_i, \tilde{m}_i)$ can be obtained by using a common ΔK_0 across the specimens. Under a stochastic approach considered here, it is not possible to derive a closed-form expression of ΔK_0 as a function of a_j's, and a numerical procedure must be applied to search for the ΔK_0 such that the covariance between the MLE of $\widehat{C}$ and $\widehat{m}$ is zero. Since the covariance of $\widehat{C}$ and $\widehat{m}$ depends on the true values of the corresponding unknown parameters, an iterative procedure is required which utilizes the MLE in place of the true values.

With the scaled Paris law in (2), the model parameters now have a clear physical meaning. The parameter ξ has dimension of crack length while the parameter C has dimension of length over time, and can be viewed as the instantaneous crack growth rate in correspondence of the crack length a for which $\Delta K(a) = \Delta K_0$. The ratio $\Delta K/\Delta K_0$ is independent of load conditions, so as the load effects on the expected time to first reach a crack length a are all embedded into C. Owing to the invariance property of MLE, the estimates under the scaled Paris law are also linked to the conventional non-scaled estimates (*i.e.*, $\Delta K_0 = 1$) by the following: $\widehat{m} = \widehat{m}'$, $\widehat{C} = \widehat{C}'\ \Delta K_0^{\widehat{m}'}$ and $\widehat{\xi} = \widehat{\xi}'\ \Delta K_0^{\widehat{m}'}$.

2.3 Goodness-of-fit assessments

The stochastic model discussed here assumes that the time increments $\Delta t_j = t_{a_j} - t_{a_{j-1}}$ between the crack length increments $\Delta a_j = a_j - a_{j-1}$ are independent gamma variables with shape parameters $\Delta\eta_j = \eta(a_j) - \eta(a_{j-1})$ and common scale parameter $\varphi = \xi C^{-1}$. Using the Probability Integral Transformation $Z_j = FGa(\Delta t_j;\ \Delta\eta_j, \varphi)$, where $FGa(\cdot)$ is the gamma distribution function, the original problem is transformed into the problem of testing if a sample of independent observations comes from a standard uniform distribution. Among many statistical procedures, we considered popular ones such as (i) the Kolmogorov statistic D; (ii) the Cramer-von Mises statistic W^2; and (iii) the Anderson–Darling statistic A^2. In the present application, the MLE of the model parameters are based on a large number of experimental points, hence assumed to be close to the true parameter values, assuring the validity of testing procedure.

3 Analysis of Virkler data

The datasets by Virkler *et al.* [21] were used to assess the accuracy of the proposed stochastic model in analyzing fatigue crack growth data observed in constant-amplitude loading tests. Virkler data consist of measurements of crack length versus number of cycles collected by using $N = 68$ specimens. An Aluminum 2024-T3 alloy rectangular specimen (558.8 mm long by 152.4 mm wide) with a thickness of 2.54 mm and a center-cracked tension geometry was used throughout the tests. The initial half crack length in the analysis was selected to be 9.0 mm while the final crack length was 49.8 mm. The accumulated number of cycles for each specimen was recorded at each 0.20 mm of crack growth over the range (9.0–36.2), at 0.40 mm over the range (36.2–44.2), and at 0.80 mm over the range (44.2–49.8). The load range was 18.69 kN, with a stress ratio 0.2. The stress-intensity factor range $\Delta K(a)$ was computed as

$$\Delta K(a) = \frac{\Delta P}{B}\sqrt{\frac{\pi\alpha}{2W}\sec\left(\frac{\pi\alpha}{2}\right)} \tag{17}$$

where B and W are specimen thickness and width, respectively, and $\alpha = 2a/W$ $(a < 0.95)$. Figure 1 below illustrates the evolution of the observed cycle size as a function of the crack length and ΔK of all 68 specimens. The solid red line is the mean behavior and dotted red lines are one-standard deviation envelope of the mean.

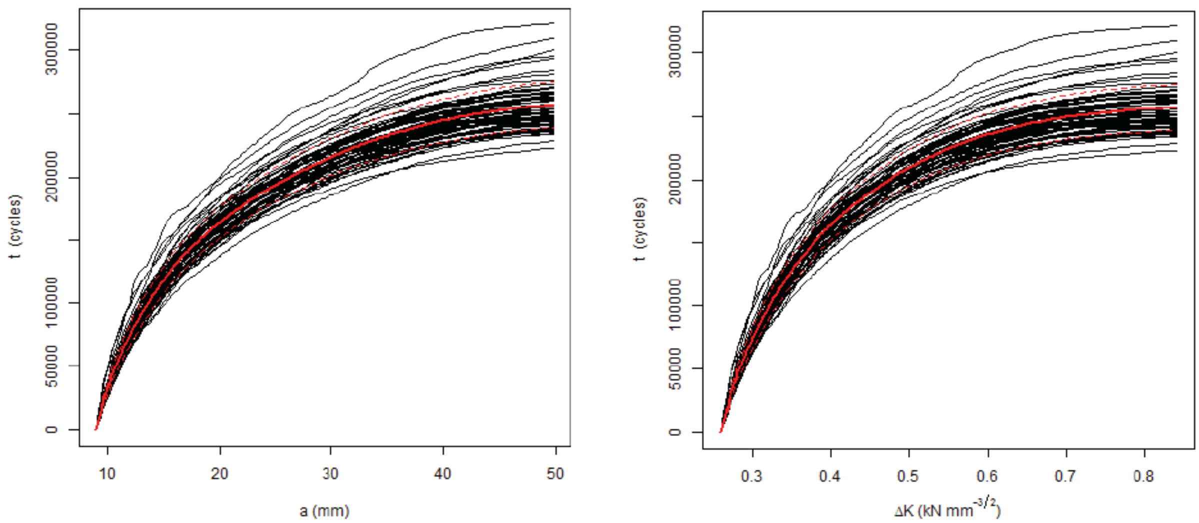

Figure 1: Plots of the cycle size versus the crack length and ΔK

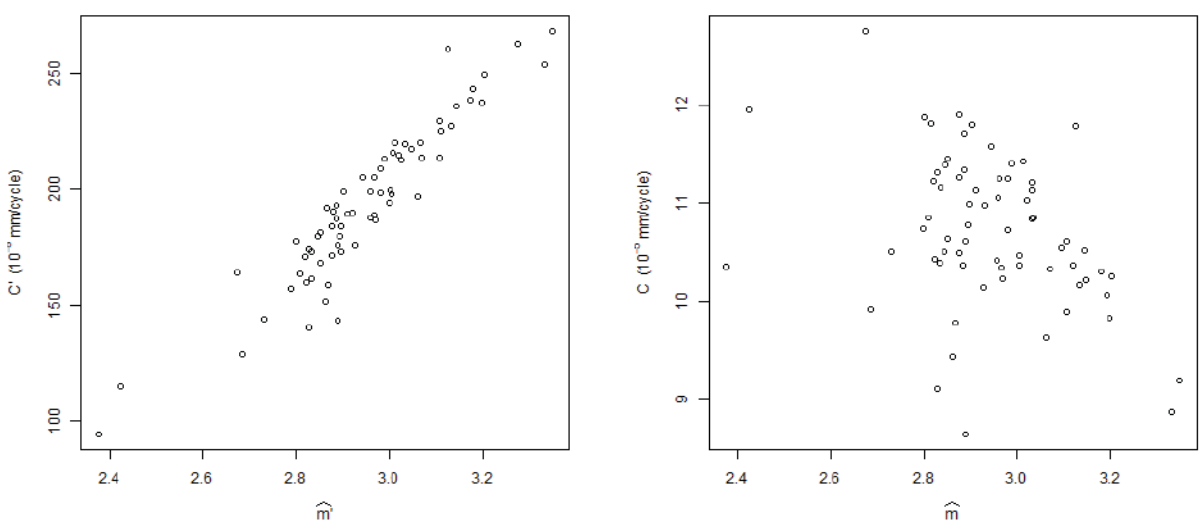

Figure 2: Conventional and individually decorrelated MLE of C versus m

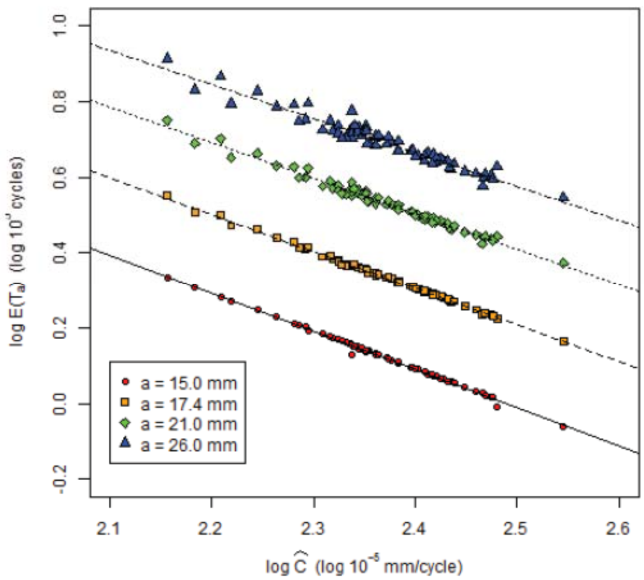

Figure 3: The time to reach a crack length of 15.0, 17.4, 21.0 and 26.0 mm versus decorrelated MLE of C in log scales

First, the MLE of the parameters C', m' and ξ' for each individual specimen were obtained under the conventional Paris regime. As expected, a very strong correlation exists between the MLE. To obtain decorrelated MLE of the parameters C and m for each individual specimen, the scaled Paris law formulation in (2) was considered. Using an iterative procedure, individually decorrelating factors were derived such that the covariance $Cov(\widehat{C}_i, \widehat{m}_i)$ of matrix $\Sigma = \mathbf{I}_n^{-1}$ given in appendix is null for each specimen. Then, based on the individual values, an average decorrelating factor $\Delta K_0 = 0.3767$ was computed. Dividing this by the load range ΔP, it was obtained $\alpha = 0.0202$. Using the scaled Paris law in (2) with this decorrelating constant, the MLE $\widehat{C}$, $\widehat{m}$ and $\widehat{\xi}$ were then computed for each single specimen. Figure 2 shows the plots of conventional and individually decorrelated MLE of C and m.

In Figure 3, the decorrelated MLE $\widehat{C}_i$ $(i = 1, \ldots, 68)$ for each specimen are plotted against the observed times to reach the crack lengths of 15.0, 17.4, 21.0 and 26.0 mm, in log scales for both axes. It appears that the MLE $\widehat{C}_i$ and the observed times to reach the crack lengths is inversely proportional to each other through a proportionality constant independent of load conditions for each crack length. To interpret these results, it is recalled that $\widehat{C}^{-1} \int_{a_0}^{a} [\Delta K_s / \Delta K(u)]^{\widehat{m}} du$ is the MLE of the expected time to reach the crack length a, and specifically, the expected time to reach the maximum observed crack length a_M is always estimated by the observed time t_{a_M}. This suggests that in case of individually decorrelated MLE, the integral $I(a;\widehat{m}_{il}) = \int_{a_0}^{a} [\Delta K_{0l}/\Delta K(u)]^{\widehat{m}_{il}} \mathrm{d}u = \int_{a_0}^{a} \left(\alpha/Y(u)\right)^{\widehat{m}_{il}} \mathrm{d}u$ depends very weakly on the estimate $\widehat{m}_i$. As a result, the ratio of mean times to reach a given crack length a is approximately equal to the inverse ratio of the corresponding parameters C, independent of load conditions. It supports the model formulation in (2), which factorizes to the product of two terms: one (C) depending on load conditions and the other ($I(a;\widehat{m})$) depending only on a. This is in favor of decorrelated estimates when analyzing fatigue crack growth data because the time to reach a given crack length is strongly depending on the current value of C.

3.1 Goodness-of-fit assessments

The goodness-of-fit of the model was analyzed using the probability integral transformation. The random quantities $Z_j = FGa(\Delta t_j;\ \Delta\eta_j, \varphi)$ were calculated across each individual specimen. Then, the ordered $Z_{(j)}$ were compared to the empirical estimates of the distribution function computed for the time increment Δt_j, and the Anderson–Darling statistic A^2 was computed. The level of significance was set at 0.10 for rejecting the null hypothesis of standard uniform distribution. With the estimated p-value of 0.26471, it was concluded that no significant evidence against the gamma process for the time to reach a given crack length was found.

The correlation coefficient between Z_j and the initial time $t_{a_{j-1}}$, and the correlation coefficient between Z_j and the initial crack length a_{j-1}, associated to each interval Δt_j, were calculated for each specimen path. The 95% confidence interval for the correlation coefficient between Z_j and $t_{a_{j-1}}$ was (-0.112, 0.194), and the null hypothesis of $\rho = 0$ was not rejected at the significance level of 0.05. The 95% confidence interval for the correlation coefficient between Z_j and a_{j-1} was (-0.037, 0.264), and the null hypothesis of $\rho = 0$ was not rejected at the significance level of 0.05. Again, no significant evidence against the proposed model was found.

3.2 Random effect for between-specimen variability

Table 1 below shows the sample statistics of the decorrelated MLE of $\widehat{C}$, $\widehat{m}$ and $\widehat{\xi}$ along with their asymptotic standard deviations. Comparing asymptotic and sampling standard deviations, we see that the parameters C and ξ are to be considered as random quantities across specimens. Ratios of sample to asymptotic standard deviations is 3.82 for C and 3.83 for ξ. For the case of m, the ratio is 2.73. Hence, a formal statistical procedure for testing the hypothesis of a common parameter across specimens was used based on the likelihood ratio test. The null hypothesis of a common m against the alternative hypothesis of different m_i values was tested by using the likelihood ratio statistic

Table 1: Sample statistics of the decorrelated MLE ($\widehat{C}$ in mm per 10^5 cycles; $\widehat{\xi}$ in mm)

Sample Statistics	$\widehat{C}$	$\widehat{m}$	$\widehat{\xi}$
Mean	10.723	2.938	0.008384
Standard deviation (SD)	0.717	0.167	0.003537
Coefficient of variation (%)	6.68	5.69	42.18
Minimum	8.540	2.376	0.004624
Maximum	12.323	3.348	0.026264
Ratio of max to min	1.44	1.41	5.67
Asymptotic SD	0.187	0.061	0.000922
Ratio of sample to asymptotic SD	3.82	2.73	3.83

$\Lambda = -2\left[\sum_{i=1}^{N} l_i(\widehat{\xi}_i, \widehat{C}_i, \widehat{m}) - \sum_{i=1}^{N} l_i(\widehat{\xi}_i, \widehat{C}_i, \widehat{m}_i)\right]$, where $l_i(\cdot)$ denotes the log-likelihood of the specimen i under the specified hypothesis. This is asymptotically distributed as χ^2 with $\nu = N - 1$ degrees of freedom. Maximizing the log-likelihood of data under both hypotheses and computing the Λ statistic, a strong evidence against the null hypothesis of a common m was found. Thus, all the parameters of the proposed model should be in principle considered as random variables across specimens. However, the real question is to what extent the variability of each parameter affects the variability of the observed quantity of interest, which is the time to first reach a given crack length a. In this regard, m acts on the expected value and variance of T_a only through the integral $\int_{a_0}^{a} [\Delta K_0/\Delta K(u)]^m \mathrm{d}u$, whose value depends very weakly on m in case of individually decorrelated MLE. On the other hand, ξ should have a negligible effect on determining the variability of T_a across the specimens since only the expected value of ξ would contribute to the marginal variance of T_a. Therefore, to explain the variability across specimens, it seems reasonable to use the model with fixed ξ and m but gamma distributed $Gamma(C;\ \gamma^{-1}, \delta)$ parameter C across specimens. Using this model, the MLE $\widehat{\xi}$, $\widehat{m}$, $\widehat{\gamma}$ and $\widehat{\delta}$ were computed and used to make inference and prediction about several quantities of interest.

Comparison between MLE and sample estimates of mean of T_a is given as function of the crack length a in Figure 4. The slight underestimation of the sample mean for small crack sizes is due to the non-Paris behavior of crack growth rate in that region. For moderate and large crack sizes, the difference between the predicted and observed mean values of T_a is within $\pm 1\%$. Also, the observed and estimated values of the variance of T_a agreed quite well.

The MLE of T_a corresponding to crack lengths 15.0, 17.4, 21.0 and 26.0 mm, are given along

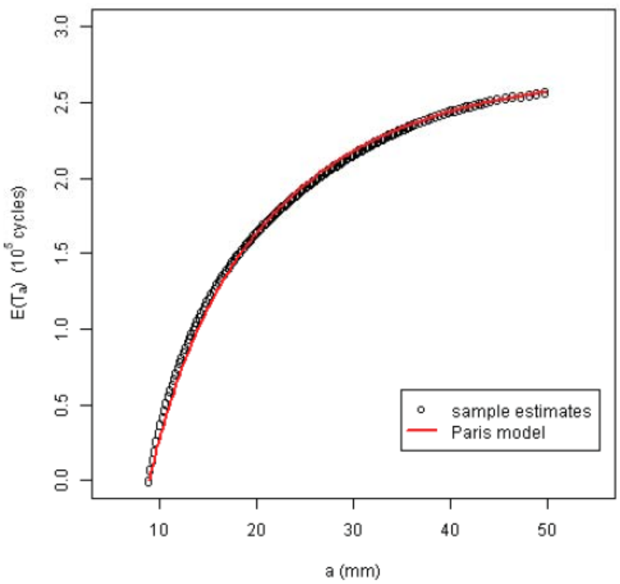

Figure 4: Sample and MLE of the mean of T_a as a function of the crack length a

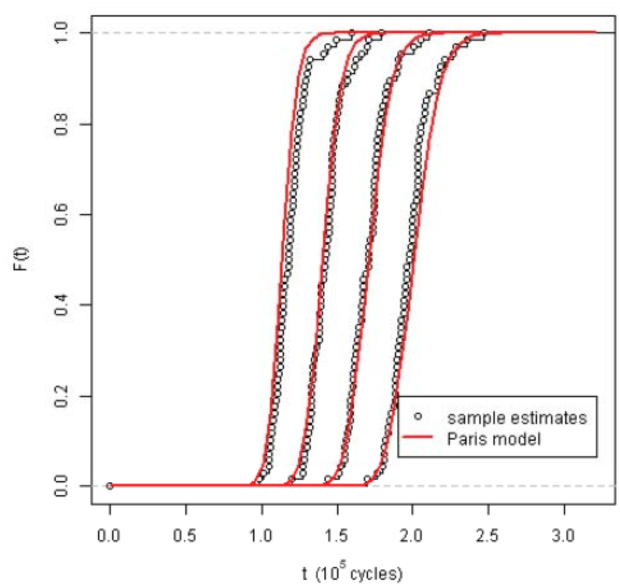

Figure 5: Sample and MLE of the distribution function of T_a for $a = 15.0$, 17.4, 21.0 and 26.0 mm

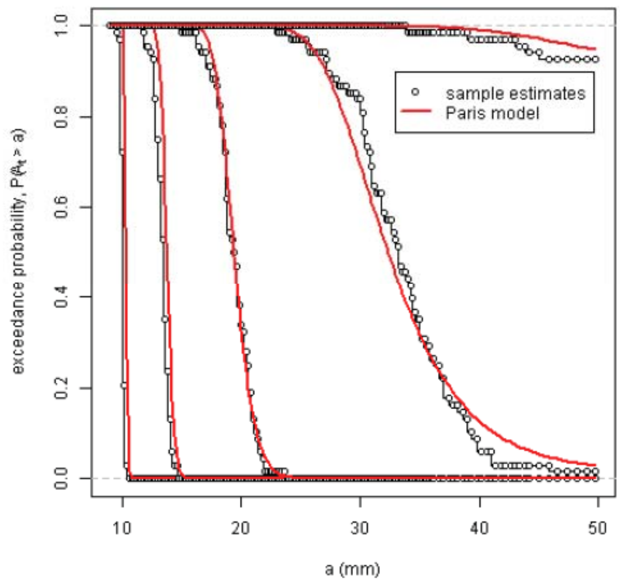

Figure 6: Sample and MLE of the exceedance probability for time values 0.34, 0.96, 1.58, 2.25 and 2.87 cycles per 10^5

with the corresponding sample estimates in Figure 5 while Figure 6 describes the MLE of the crack exceedance probability curves corresponding to the time values 0.34, 0.96, 1.58, 2.25 and 2.87 cycles per 10^5, along with the corresponding sample estimates. Both plots show good fits of the model. The MLE and the sample estimates of (10 (20) 90) percentiles of the distribution of T_a are shown as a function of the crack length a in Figure 7. The sample estimate of the pth quantile is obtained as $Q(p) = (1-f)x_j + f\ x_{j+1}$ where $j = p\ (N+1)$, $f = \lfloor p\ (N+1) \rfloor - j$, and x_j is the j-th order statistic, if $1 \leq j \leq N$. Again, underestimation is observed at small crack sizes due to non-Paris behavior of crack growth rate in that region. For moderate and large crack size values, the difference between predicted and observed values is within $\pm 3\%$.

The hypothesis of gamma distribution for the hyperparameter C was tested using the Anderson–Darling A^2 statistic, and this hypothesis could not be rejected at the significance level of 0.10. Figure 8 below depicts the empirical and theoretical densities and distributions as well as the Q-Q plot and P-P plot based on the probability integral transformation of ordered observations of C under gamma distribution. The fit of gamma model to observed data appears to be good.

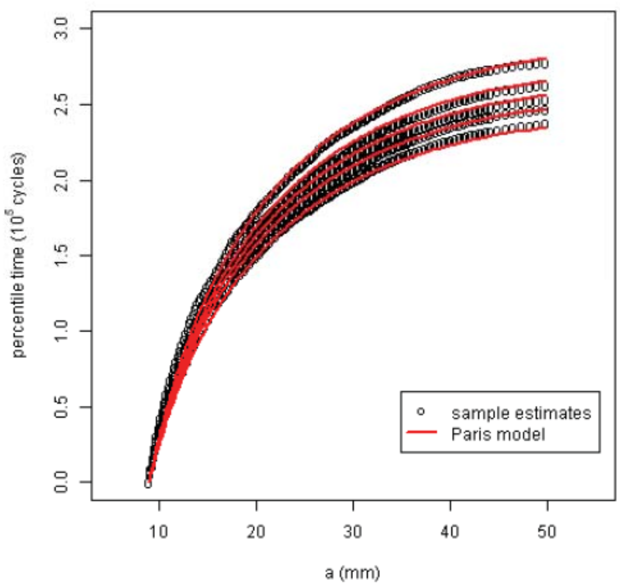

Figure 7: Sample and MLE of (10 (20) 90) percentiles of T_a as a function of the crack length a

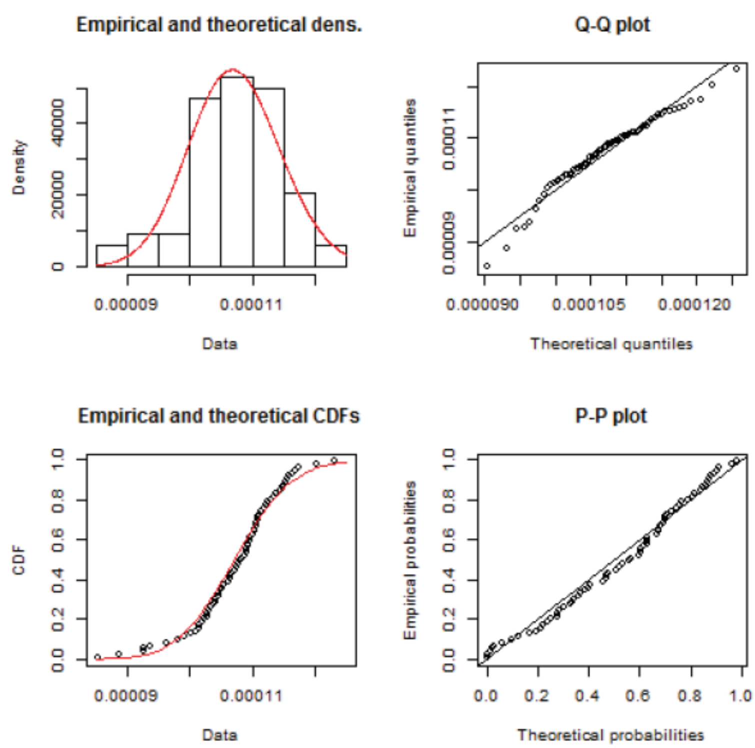

Figure 8: Diagnostic plots of gamma distribution for the hyperparameter C

4 Conclusions

In this work, the time to reach a given crack size a was directly modeled as a random process over the crack size domain. The advantages of this approach are: (1) since the time to reach any crack size is a random process with independent increments, its distribution is completely defined; (2) the process mean matches with the conventional deterministic crack growth rate model (*viz.*, Paris law); and (3) population heterogeneity can be explained by using a random effect, still leading to analytical solutions for the marginal distribution of the time to reach any given crack size as well as its mean and variance. The adequacy of the gamma process in analyzing and interpreting the crack growth under constant amplitude cyclic loading was assessed by using the experimental datasets from Virkler *et al.* [21]. By incorporating a suitable scale factor ΔK_0 into the crack growth rate model, the quasi-decorrelated MLE of the parameters could be obtained. It was observed that the scaling factor is proportional to the load range and does not depend on the stress ratio. Also, it was observed that the decorrelated MLE of C for each specimen are perfectly correlated to the observed time to reach the maximum crack length and are strongly correlated to the observed time to reach a given crack size a. Subsequently, a random effects model based on the decorrelated MLE was constructed where C is assumed to be a gamma random variable. The mean and variance of the time T_a to reach a given crack length a, the probability distribution of T_a corresponding to the selected values of a, the exceedance probability corresponding to the selected cycle times, and the percentiles of the distribution of T_a as a function of a were investigated under the random effects model, and all supported the adequacy of the model in describing the crack growth evolution.

As a future work, it is desired to improve the proposed stochastic model to analyze the crack growth data not completely belonging to Region II regime. Since the crack growth rate behaves non-linearly at the borders of Region II, a modified (deterministic) model based on the Paris law will be considered for the stochastic process mean, and its performance will be compared to that based on the scaled Paris model considered in this work. A further model validation is also sought using the datasets from Ghonem and Dore (1985).

5 References

1. ASTM E647–13ae1. Standard test method for measurement of fatigue crack growth rates.

2. BS 6835–1:1998. Method for the determination of the rate of fatigue crack growth in metallic materials. Fatigue crack growth rates of above 10–8 m per cycle.

3. Casciati F, Colombi P, Faravelli L. Fatigue crack size probability distribution via a filter technique. Fatigue Fract Engng Mater Struct 1992;15:463–75.

4. Casciati F, Colombi P, Faravelli L. Inherent variability of an experimental crack growth curve. Struct Saf 2007;29:66–76.

5. Casciati F, Faravelli L. Fragility analysis of complex structural system. Research Studied Press; 1991.

6. Colombi P, Faravelli L. Stochastic finite elements via response surface: fatigue crack growth problems. In: Guedes Soares C, editor. Probabilistic methods for structural design. Kluwer Academic Publishers; 1997. p. 313–38.

7. Ditlevsen O. Random fatigue crack growth – a first passage problem. Engng Fract Mech 1986;23:467–77.

8. Ditlevsen O, Olesen R. Statistical analysis of the Virkler data on fatigue crack growth. Engng Fract Mech 1986;25:177–95.

9. Ditlevsen O, Sobczyk K. Random fatigue crack growth with retardation. Engng Fract Mech 1986;24:861–78.

10. Kloeden PE, Platen E. Numerical solution of stochastic differential equations. Berlin: Springer; 1995.

11. Kozin F, Bogdanoff JL. A critical analysis of some probabilistic models of fatigue crack growth. Engng Fract Mech 1981;14:59–89.

12. Lin YK, Yang JN. A stochastic theory of fatigue crack propagation. AIAA J 1985;23:117–24.

13. Madsen HO. Probabilistic and deterministic models for predicting damage accumulation due to time varying loading. Dialog 5–82, Lyngby (Denmark):Danish Engineering Academy; 1983.

14. Ortiz K, Kiremidjian AS. Time series analysis of fatigue crack growth rate data. Engng Fract Mech 1986;24:657–75.

15. Ortiz K, Kiremidjian AS. Stochastic modeling of fatigue crack growth. Engng Fract Mech 1988;29:317–34.

16. Ostergaard DF, Hillberry BM. Characterization of variability in fatigue crack propagation data. In: Probabilistic methods for design and maintenance of structures, STP-798, Philadelphia: ASTM; 1983, p. 97–115.

17. Sobczyk K. Modelling of random fatigue crack growth. Engng Fract Mech 1986;24:609–23.

18. Sobczyk K, Trebicki J. Modelling of random fatigue by cumulative jump processes. Engng Fract Mech 1989;34:477–93.

19. Sobczyk K, Trebicki J. Cumulative jump-correlated model for random fatigue. Engng Fract Mech 1991;40:201–10.

20. Tang J, Spencer Jr BF. Reliability solution for fatigue crack growth problem. Engng Fract Mech 1989;34:419–33.

21. Virkler DA, Hillberry BM, Goel PK. The statistical nature of fatigue crack propagation. AFFDL-TR-78-43; 1978.

22. Wu WF, Ni CC. Probabilistic models of fatigue crack propagation and their experimental verification. Probab Engng Mech 2004;19:247–57.

23. Wu WF, Ni CC. Statistical aspects of some fatigue crack growth data. Engng Fract Mech 2007;74:2952–63.

24. Yang J, Manning S. A simple second order approximation for stochastic crack growth analysis. Engng Fract Mech 1996;53:677–86.

6 Appendix

6.1 Maximum likelihood estimation for a single specimen path

The log likelihood function based on (7) is

$$\log L(C, m, \xi|\text{data}) = -\sum_{j=1}^{M} \log \Gamma(\Delta\eta_j) - \eta(a_M) \log(\xi C^{-1}) + \sum_{j=1}^{M} (\Delta\eta_j - 1) \log \Delta t_j - \xi^{-1} C t_{a_M} \quad (6.1)$$

By equating the first partial derivatives of (6.1) with respect to each parameter to zero, we obtain

$$\widehat{C} = I_{1M}(\widehat{m})/t_{a_M}, \tag{6.2}$$

$$\sum_{j=1}^{M} I_{1j}(\widehat{m})\Big[\psi\big(\widehat{\xi}^{-1}I_{1j}(\widehat{m})\big) - \log\Delta t_j\Big] = I_{1M}(\widehat{m})\log\big(\widehat{\xi}^{-1}I_{1M}(\widehat{m})/t_{a_M}\big), \tag{6.3}$$

$$\sum_{j=1}^{M} I_{2j}(\widehat{m})\Big[\psi\big(\widehat{\xi}^{-1}I_{1j}(\widehat{m})\big) - \log\Delta t_j\Big] = I_{2M}(\widehat{m})\log\big(\widehat{\xi}^{-1}I_{1M}(\widehat{m})/t_{a_M}\big) \tag{6.4}$$

where

$$\begin{aligned}
I_{1j}(\widehat{m}) &= \int_{a_{j-1}}^{a_j} [\Delta K_0/\Delta K(u)]^{\widehat{m}}\mathrm{d}u,\\
I_{1M}(\widehat{m}) &= \int_{a_0}^{a_M} [\Delta K_0/\Delta K(u)]^{\widehat{m}}\mathrm{d}u,\\
I_{2j}(\widehat{m}) &= \int_{a_{j-1}}^{a_j} [\Delta K_0/\Delta K(u)]^{\widehat{m}}\log\big(\Delta K_0/\Delta K(u)\big)\mathrm{d}u,\\
I_{2M}(\widehat{m}) &= \int_{a_0}^{a_M} [\Delta K_0/\Delta K(u)]^{\widehat{m}}\log\big(\Delta K_0/\Delta K(u)\big)\mathrm{d}u,
\end{aligned}$$

and $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$ is the digamma function. The MLE $\widehat{m}$ and $\widehat{\xi}$ are obtained by numerically solving the non-linear system of (6.3) and (6.4), and then $\widehat{C}$ is obtained subsequently from (6.2).

The elements of the Fisher information matrix $\mathbf{I}_n$ are

$$\begin{aligned}
-E\left(\frac{\partial^2\log L}{\partial C^2}\right) &= J_{11} = C^{-2}\xi^{-1}I_{1M}(m),\\
-E\left(\frac{\partial^2\log L}{\partial m^2}\right) &= J_{22} = \xi^{-2}\sum_{j=1}^{M} I_{2j}^2(m)\psi'\big(\xi^{-1}I_{1j}(m)\big),\\
-E\left(\frac{\partial^2\log L}{\partial \xi^2}\right) &= J_{33} = \xi^{-4}\sum_{j=1}^{M} I_{1j}^2(m)\psi'\big(\xi^{-1}I_{1j}(m)\big) - \xi^{-3}I_{1M}(m),\\
-E\left(\frac{\partial^2\log L}{\partial C\partial m}\right) &= J_{12} = J_{21} = -C^{-1}\xi^{-1}I_{2M}(m),\\
-E\left(\frac{\partial^2\log L}{\partial C\partial \xi}\right) &= J_{13} = J_{31} = 0,\\
-E\left(\frac{\partial^2\log L}{\partial m\partial \xi}\right) &= J_{23} = J_{32} = -\xi^{-3}\sum_{j=1}^{M} I_{1j}(m)I_{2j}(m)\psi'\big(\xi^{-1}I_{1j}(m)\big) + \xi^{-2}I_{2M}(m),
\end{aligned}$$

where $\psi'(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\Gamma(z)$ is the trigamma function. The elements of the variance-covariance matrix

$\Sigma = \mathbf{I}_n^{-1}$ are then

$$\begin{aligned}
Var(\widehat{C}) &= (J_{22}J_{33} - J_{23}^2)/\det(\mathbf{I}_n),\\
Var(\widehat{m}) &= (J_{11}J_{33})/\det(\mathbf{I}_n),\\
Var(\widehat{\xi}) &= (J_{11}J_{22} - J_{12}^2)/\det(\mathbf{I}_n),\\
Cov(\widehat{C},\widehat{m}) &= -(J_{12}J_{33})/\det(\mathbf{I}_n),\\
Cov(\widehat{C},\widehat{\xi}) &= (J_{12}J_{23})/\det(\mathbf{I}_n),\\
Cov(\widehat{m},\widehat{\xi}) &= -(J_{11}J_{23})/\det(\mathbf{I}_n)
\end{aligned}$$

where $\det(\mathbf{I}_n) = J_{11}J_{22}J_{33} - J_{11}J_{23}^2 - J_{33}J_{12}^2$.

6.2 Maximum likelihood estimation for the random effects model

The log likelihood function for the random effects model is

$$\begin{aligned}
\log L(m,\xi,\delta,\gamma|\text{data}) &= \sum_{i=1}^{N}\log L_i(m,\xi,\delta,\gamma|\text{data}_i)\\
&= N\delta\log(\xi\gamma) + \sum_{i=1}^{N}\log\Gamma\big(\eta_i(a_{M_i})+\delta\big) - N\log\Gamma(\delta) - \sum_{i=1}^{N}\sum_{j=1}^{M_i}\log\Gamma(\Delta\eta_{ij})\\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{M_i}(\Delta\eta_{ij}-1)\log\Delta t_{ij} - \sum_{i=1}^{N}\big(\eta_i(a_{M_i})+\delta\big)\log(t_{iM_i}+\xi\gamma) \qquad (6.5)
\end{aligned}$$

Here, $M_i = M$, $\Delta\eta_{ij} = \Delta\eta_j$, and $\eta_i(a_{M_i}) = \eta(a_M)$ for $i = 1,\ldots,N$. By computing the first partial derivatives of (6.5) with respect to each parameter and equating them to zero, we obtain

$$\widehat{\delta} = \frac{I_{1M}(\widehat{m})}{N\Big(\widehat{\gamma}\sum_{i=1}^{N}\big(\widehat{\xi}\widehat{\gamma}+t_{iM}\big)^{-1}\Big)^{-1} - \widehat{\xi}}, \qquad (6.6)$$

$$N^{-1}\sum_{i=1}^{N}\log(\widehat{\xi}\widehat{\gamma}+t_{iM}) - \log(\widehat{\xi}\widehat{\gamma}) = \psi\big(\widehat{\delta}+\widehat{\xi}^{-1}I_{1M}(\widehat{m})\big) - \psi(\widehat{\delta}), \qquad (6.7)$$

$$\sum_{i=1}^{N}\sum_{j=1}^{M}I_{1j}(\widehat{m})\Big[\log\Delta t_{ij} - \log(\widehat{\xi}\widehat{\gamma})\Big] = N\sum_{j=1}^{M}I_{1j}(\widehat{m})\Big[\psi\big(\widehat{\xi}^{-1}I_{1j}(\widehat{m})\big) - \psi(\widehat{\delta})\Big], \qquad (6.8)$$

$$I_{1M}^{-1}(\widehat{m})\sum_{i=1}^{N}\sum_{j=1}^{M}I_{1j}(\widehat{m})\Big[\psi\big(\widehat{\xi}^{-1}I_{1j}(\widehat{m})\big) - \log\Delta t_{ij}\Big] = I_{2M}^{-1}(\widehat{m})\sum_{i=1}^{N}\sum_{j=1}^{M}I_{2j}(\widehat{m})\Big[\psi\big(\widehat{\xi}^{-1}I_{1j}(\widehat{m})\big) - \log\Delta t_{ij}\Big], \qquad (6.9)$$

The MLE $\widehat{m}$, $\widehat{\xi}$, and $\widehat{\gamma}$ are computed by numerically solving the non-linear system of (6.7–6.9), and then $\widehat{\delta}$ is obtained subsequently from (6.6).