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On the Joint Distribution of Placement Statistics under Progressive Censoring and Applications to Precedence Test

N. Balakrishnan[1], Ram C. Tripathi[2], Nandini Kannan[2]

ABSTRACT

In reliability and survival analysis, comparison of two or more populations is an important problem. For example, while comparing a treatment group with a control group, one may be interested in determining whether the observations in the treatment group have a longer lifetime than those from the control group, that is, whether the treatment is effective or not. In such a study, it would be extremely valuable to make a decision based on early failures. In this paper, we consider independent progressively Type-II censored random samples from two populations with cumulative distribution function's (cdf) $F(\cdot)$ and $G(\cdot)$ respectively, and discuss a precedence test for testing the equality of the two distributions based on placements. For this purpose, we derive the joint distribution of the first l placement statistics from the progressively censored sample from $F(\cdot)$. We then derive the exact null distribution of the precedence test statistic which is simply the sum of the placements. We provide the rejection regions for fixed levels of significance and various sample sizes and different progressive censoring schemes.

Key Words: Progressive Type-II censoring, placements, precedence and exceedance statistics, nonparametric tests of homogeneity, Wilcoxon rank-sum test.

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1 Introduction

Comparison of two or more populations is a common and important problem in reliability and survival analysis. For example, while comparing a new treatment protocol with the control, one may be interested in assessing whether the observations corresponding to the treatment population have a longer lifetime than those from the control population meaning that the treatment is indeed effective. In reliability, one may also be interested in comparing two production lines to determine which of the two processes produces units with longer lifetimes. There are several parametric and nonparametric methods for comparing two populations based on independent random samples. The standard nonparametric procedures are based on ranks including Wilcoxon's rank sum-test, Mann-Whitney test, Van der Wearden test, and Mood's test. However, these tests are based on complete samples. In the presence of censoring, variations of these tests have been discussed in the literature and among these are the Cox-Mantel test and the log-rank test, for example.

In drug studies, it is often desirable to make a quick decision on the efficacy of a new drug based on early data on lifetimes. For example, if the first 5 deaths observed all belong to the control group, it might be reasonable to question at that point of time whether the treatment is effective. Such early decisions result in savings, both in terms of time and costs associated with testing.

Nelson (1963, 1993) discussed a procedure based on the number of failures from one sample that occur before a specified failure from the second sample. These counts are referred to as *precedence* statistics. Epstein (1954) had earlier considered similar test for comparing two distributions based on *exceedances*, where exceedances are the numbers of failures from the first sample that exceed failures from the second sample. Orban and Wolfe (1982) proposed a two-sample test for complete samples based on precedences (which they call placements) that generalizes Mann-Whitney's test. They proposed a linear placement statistic (similar to the general linear rank statistic) and showed that the two are equivalent for the Mann-Whitney-Wilcoxon scoring functions. The ideas of precedence and exceedance statistics have been extended by several other authors including Bairamov and Eryilmaz (2006) who discussed the exact distribution of the precedence statistic in the case when one of the samples is progressively Type-II censored. If the samples are complete, as expected, the distribution reduces to the classical negative hypergeometric distribution. Ng and Balakrishnan (2005) suggested several tests based on placement statistics when one of the samples is progressively censored. In particular, they proposed weighted precedence and maximal precedence tests and demonstrated that these procedures are more powerful than those based just on the precedence statistic. An excellent discussion of precedence tests and their variants can be found in the recent book by Balakrishnan and Ng (2006).

In this paper, we develop a procedure based on placements and precedence statistics when both samples are progressively Type-II censored. We derive the exact null distributions of these statistics and show that the joint distribution of the placement statistics reduces to the classical negative hypergeometric distribution in the case of complete samples. We provide the critical values for different sample sizes and progressive censoring schemes.

In Section 2, we describe the notation and present the main result concerning the joint distribution of the placement statistics. We show that it reduces to the negative hypergeometric distribution in the case when both samples are complete. In Section 3, we tabulate the joint probability distribution of the placement statistics and the distribution of precedence statistic for different sample sizes and progressive censoring schemes. We also present some tables of the distribution function of the precedence test statistic that can be used to determine rejection regions for testing whether the treatment is effective at a specified level of significance α.

2 Preliminaries and Notation

In this section, we derive the joint probability mass function of the placement statistics based on independent progressively Type-II censored samples from the two populations. Let $F(\cdot)$ denote the cdf of the control group and $G(\cdot)$ the cdf of the treatment group, with both $F(\cdot)$ and $G(\cdot)$ being continuous.

In conventional Type-I and Type-II censoring schemes, we do not allow for removal of units at points other than the terminal point of the experiment. We discuss here a more general censoring scheme called progressive censoring that allows for units to be removed at certain times during the experiment in addition to removing some units at the terminal point. Such progressive censoring schemes are very appealing in reliability and life-testing experiments since decisions regarding acceptance and rejection of hypotheses can be made having observed only a few early failures from the two samples. Ng and Balakrishnan (2005) have mentioned that such censoring schemes are especially useful when (i) life tests involve expensive experimental units, since the units that have not failed can still be used for some other purposes, and (ii) reliable decisions can be made early in the experiment thus resulting in saving of time and cost. For an elaborate review of various developments on progressive censoring, one may refer to the book by Balakrishnan and Aggarwala (2000).

Consider an experiment in which n_1 units from the control group and n_2 units from the treatment group are placed simultaneously on a life-test. When the first failure in the control group occurs, R_1 units are randomly removed from the remaining $n_1 - 1$ surviving units. At the second failure, R_2 units from the remaining $n_1 - 2 - R_1$ units are randomly removed. The test continues until the m_1^{th} failure. At this time, all remaining $R_{m_1} = n_1 - m_1 - R_1 - R_2 - \cdots - R_{m_1-1}$ units are removed. The R_i's are fixed prior to the study. If $R_1 = R_2 = \cdots = R_{m_1} = 0$, we have $n_1 = m_1$ which corresponds to the complete sample for the control group. If $R_1 = R_2 = \cdots = R_{m_1-1} = 0$, then $R_{m_1} = n_1 - m_1$ which corresponds to the conventional Type-II right censoring scheme. We similarly observe a progressively Type-II censored sample from the treatment group with m_2 being the number of complete failures and $(S_1, \cdots, S_{m_2})$ being the corresponding progressive censoring scheme. With $\boldsymbol{R} = (R_1, R_2, \cdots, R_{m_1})$ and $\boldsymbol{S} = (S_1, S_2, \cdots, S_{m_2})$, let us denote the m_1 failures from the control group by $\boldsymbol{X} = (X_{1:m_1:n_1}^{(\boldsymbol{R})}, \cdots, X_{m_1:m_1:n_1}^{(\boldsymbol{R})})$ and the m_2 failures from the treatment group by $\boldsymbol{Y} = (Y_{1:m_2:n_2}^{(\boldsymbol{S})}, \cdots, Y_{m_2:m_2:n_2}^{(\boldsymbol{S})})$. We refer to these as the X-sample and the Y-sample, respectively.

Our aim is to test the hypothesis that the two groups are the same against the hypothesis that the treatment group has longer lifetime than the control group, i.e., we wish to test

$$H_0 \;:\; F(x) = G(x) \;\; \forall x$$

against the alternative

$$H_1 \;:\; F(x) \geq G(x)$$

with strict inequality holding for at least one x. The general alternative hypothesis specified here includes the location-shift as well as the Lehmann-type alternatives as special cases.

3 Placement Statistics and Joint Distribution

We define the placement statistics as follows: the i^{th} placement from the X-sample, denoted by U_i, is precisely the number of X-failures that fall between the $(i-1)^{th}$ and the i^{th} Y-failures. Formally, for a fixed i, we have $U_i = \#$ of $X^{(\boldsymbol{R})}_{j:m_1:n_1}$ such that $Y^{(\boldsymbol{S})}_{i-1:m_2:n_2} < X^{(\boldsymbol{R})}_{j:m_1:n_1} < Y^{(\boldsymbol{S})}_{i:m_2:n_2}$, $j = 1, 2, \cdots, m_1$.

We wish to determine the joint probability mass function (pmf) of $U_1, U_2, \cdots, U_l$, given by

$$P(U_1 = u_1, \cdots, U_l = u_l) = \int \cdots \int_{\Omega} \cdots \int f_{\boldsymbol{X}}(\boldsymbol{x}) \;\; dx_1 \cdots dx_{m_1}, \tag{1}$$

where the region of integration Ω is

$$\begin{aligned} \Omega = \Big\{ (x_1, \cdots, x_{m_1}) : \quad & -\infty < x_1 < \cdots < x_{u_1} < y_1, \\ & y_1 < x_{u_1+1} < \cdots < x_{u_{(2)}} < y_2, \\ & \cdots, y_{l-1} < x_{u_{(l-1)}+1} < \cdots < x_{u_{(l)}} < y_l, \\ & y_l < x_{u_{(l)}+1} < \cdots < x_{m_1} < \infty \Big\} \end{aligned}$$

with $u_{(0)} = 0$ and

$$u_{(i)} = u_1 + \cdots + u_i \text{ for } i = 1, 2, \cdots, l. \tag{2}$$

The joint pdf of $\boldsymbol{X} = (X^{(\boldsymbol{R})}_{1:m_1:n_1}, \cdots, X^{(\boldsymbol{R})}_{m_1:m_1:n_1})$ is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \; n_1(n_1 - R_1 - 1) \cdots (n_1 - R_1 - \cdots - R_{m_1-1} - m_1 + 1)$$

$$\times f(x_1)[1-F(x_1)]^{R_1}\cdots f(x_{m_1})[1-F(x_{m_1})]^{R_{m_1}},$$

$$-\infty < x_1 < \cdots < x_{m_1} < \infty; \tag{3}$$

see Balakrishnan and Aggarwala (2000) for details. To simplify the notation, we denote

$$A = n_1(n_1 - R_1 - 1)\cdots(n_1 - R_1 - \ldots - R_{m_1-1} - m_1 + 1). \tag{4}$$

Evidently, for obtaining the distribution of the placements $U_1, \cdots, U_l$, we need to evaluate the multiple integral in (1) which may be evaluated in pieces as follows. Consider the first integral

$$\begin{aligned} I_1(y_1) = \quad & \int \cdots \int_{-\infty<x_1<\cdots<x_{u_1}<y_1} \cdots \int f(x_1)[1-F(x_1)]^{R_1}\cdots \\ & \times f(x_{u_1})[1-F(x_{u_1})]^{R_{u_1}} dx_1 \cdots dx_{u_1}. \end{aligned} \tag{5}$$

Let us denote the partial sum from the left as

$$R_{(i)} = R_1 + \cdots + R_i \tag{6}$$

and the partial sum from the right as

$$R_{[i]} = R_{m-i+1} + \cdots + R_m. \tag{7}$$

The integral in (5) then simplifies to

$$I_1(y_1) \;\; = \;\; \sum_{i_1=0}^{u_1} \gamma_{i_1,u_1}(R_1,\cdots,R_{u_1})\,[1-F(y_1)]^{\sum_{k_1=u_1-i_1+1}^{u_1}(R_{k_1}+1)}, \tag{8}$$

where

$$\begin{aligned} \gamma_{i_j,u_j}(R_{u_{(j-1)}+1},\cdots,R_{u_{(j)}}) = \quad & \frac{(-1)^{i_j}}{\left\{\prod_{g=1}^{i_j}\sum_{k=u_j-i_j+1}^{u_j-i_j+g}(R_{u_{(j-1)}+k}+1)\right\}} \\ & \times\frac{1}{\left\{\prod_{g=1}^{u_j-i_j}\sum_{k=g}^{u_j-i_j}(R_{u_{(j-1)}+k}+1)\right\}}. \end{aligned} \tag{9}$$

Similarly, the second integral

$$I_2(y_1,y_2) = \int\cdots\int_{y_1<x_{u_1+1}<\cdots<x_{u_{(2)}}<y_2}\cdots\int f(x_{u_1+1})[1-F(x_{u_1+1})]^{R_{u_1+1}}\cdots \times f(x_{u_{(2)}})[1-F(x_{u_{(2)}})]^{R_{u_{(2)}}+1}dx_{u_1+1}\cdots dx_{u_{(2)}}$$

simplifies to

$$\sum_{i_2=0}^{u_2}\gamma_{i_2,u_2}(R_{u_1+1},\cdots,R_{u_{(2)}})\,[1-F(y_2)]^{\sum_{k_2=u_2-i_2+1}^{u_2}(R_{u_1+k_2}+1)}\,[1-F(y_1)]^{\sum_{k_2=1}^{u_2-i_2}(R_{u_1+k_2}+1)}. \tag{10}$$

Proceeding in an analogous manner, the last two integrals are obtained as follows:

$$\begin{aligned} I_l(y_{l-1},y_l) = & \int\cdots\int_{y_{l-1}<x_{u_{(l-1)}+1}<\cdots<x_{u_{(l)}}<y_l}\cdots\int f(x_{u_{(l-1)}+1})[1-F(x_{u_{(l-1)}+1})]^{R_{u_{(l-1)}+1}} \\ & \times\cdots f(x_{u_{(l)}})[1-F(x_{u_{(l)}})]^{R_{u_{(l)}}+1}dx_{u_{(l-1)}+1}\cdots dx_{u_{(l)}} \\ = & \sum_{i_l=0}^{u_l}\gamma_{i_l,u_l}(R_{u_{(l-1)}+1},\cdots,R_{u_{(l)}})\,[1-F(y_l)]^{\sum_{k_l=u_l-i_l+1}^{u_l}(R_{u_{(l-1)}+k_l}+1)} \\ & \times[1-F(y_{l-1})]^{\sum_{k_l=1}^{u_l-i_l}(R_{u_{(l-1)}+k_l}+1)} \end{aligned} \tag{11}$$

and

$$\begin{aligned} I_{l+1}(y_l) = & \int\cdots\int_{y_l<x_{u_{(l)}+1}<\cdots<x_{m_1}<\infty}\cdots\int f(x_{u_{(l)}+1})[1-F(x_{u_{(l)}+1})]^{R_{u_{(l)}+1}} \\ & \times\cdots f(x_{m_1})[1-F(x_{m_1})]^{R_{m_1}}dx_{u_{(l)}+1}\cdots dx_{m_1} \\ = & \sum_{i_{l+1}=0}^{m_1-u_{(l)}}\gamma_{i_{l+1},u_{l+1}}(R_{u_{(l)}+1},\cdots,R_{m_l})\,[1-F(\infty)]^{\sum_{k_{l+1}=u_{l+1}-i_{l+1}+1}^{u_{l+1}}(R_{u_{(l)}+k_{l+1}}+1)} \\ & \times[1-F(y_l)]^{\sum_{k_{l+1}=1}^{u_{l+1}-i_{l+1}}(R_{u_{(l)}+k_{l+1}}+1)} \\ = & \gamma_{0,u_{l+1}}(R_{u_{(l)}+1},\cdots,R_{m_l})\,[1-F(y_l)]^{\sum_{k_{l+1}=1}^{u_{l+1}}(R_{u_{(l)}+k_{l+1}}+1)}. \end{aligned} \tag{12}$$

From eqs. (8), (10), (11), and (12), we obtain

$$
\begin{aligned}
J(y_1,\cdots,y_l) = \quad & I_1(y_1)I_2(y_1,y_2)\cdots I_l(y_{l-1},y_l)I_{l+1}(y_l) \\
= \quad & \sum_{i_1=0}^{u_1}\cdots\sum_{i_l=0}^{u_l}\prod_{j=1}^{l+1}\gamma_{i_j,u_j}(R_{u_{(j-1)}+1},\cdots,R_{u_{(j)}}) \\
& \times[1-F(y_1)]^{\sum_{k_1=u_1-i_1+1}^{u_1}(R_{k_1}+1)+\sum_{k_2=1}^{u_2-i_2}(R_{u_1+k_2}+1)} \\
& \times[1-F(y_2)]^{\sum_{k_2=u_2-i_2+1}^{u_2}(R_{u_1+k_2}+1)+\sum_{k_3=1}^{u_3-i_3}(R_{u_{(2)}+k_3}+1)} \\
& \times\ldots \\
& \times[1-F(y_{l-1})]^{\sum_{k_{l-1}=u_{l-1}-i_{l-1}+1}^{u_{l-1}}(R_{u_{(l-2)}+k_{l-1}}+1)+\sum_{k_l=1}^{u_l-i_l}(R_{u_{(l-1)}+k_l}+1)} \\
& \times[1-F(y_l)]^{\sum_{k_l=u_l-i_l+1}^{u_l}(R_{u_{(l-1)}+k_l}+1)+\sum_{k_l=1}^{u_{l+1}}(R_{u_{(l)}+k_l}+1)}, \\
& \qquad -\infty<y_1<\cdots<y_l<\infty, \qquad (13)
\end{aligned}
$$

where $u_{l+1}=m_1-u_{(l)}$. Hence, the conditional joint pmf of $(U_1,\cdots,U_l)$, given $(Y_{1:m_2:n_2}=y_1,\cdots,Y_{m_2:m_2:n_2}=y_{m_2})$, is simply

$$
\begin{aligned}
P(U_1=u_1,\cdots,U_l=u_l|y_1,\cdots,y_{m_2}) = A\ J(y_1,\cdots,y_l),& \\
u_i=0,1,\ldots;\ \sum_{i=1}^{l}u_i\le m_1.& \qquad (14)
\end{aligned}
$$

Next, the joint pdf of $\boldsymbol{Y}=(Y_{1:m_2:n_2}^{(\boldsymbol{S})},\cdots,Y_{m_2:m_2:n_2}^{(\boldsymbol{S})})$ is given by

$$
\begin{aligned}
f_{\boldsymbol{Y}}(\boldsymbol{y}) = \quad & n_2(n_2-S_1-1)\cdots(n_2-S_1-\cdots-S_{m_2-1}-m_2+1) \\
& \times f(y_1)[1-F(y_1)]^{S_1}\cdots f(y_{m_2})[1-F(y_{m_2})]^{S_{m_2}}, \\
& \qquad -\infty<y_1<\cdots<y_{m_2}<\infty. \qquad (15)
\end{aligned}
$$

Let us denote $B=n_2(n_2-S_1-1)\cdots(n_2-S_1-\ldots-S_{m_2-1}-m_2+1)$. Then, from eqs. (14) and (15), we have the unconditional joint pmf of $\boldsymbol{U}=(U_1,U_2,\cdots,U_l)$ as

$$
\begin{aligned}
P(U_1=u_1,\cdots,U_l=u_l) = AB\sum_{i_1=0}^{u_1}\cdots\sum_{i_l=0}^{u_l}\prod_{j=1}^{l+1}\gamma_{i_j,u_j}(R_{u_{(j-1)}+1},\cdots,R_{u_{(j)}})& \\
\times\int\ldots\int_{-\infty<y_1<\cdots<y_{m_2}<\infty}\cdots\int\prod_{i=1}^{m_2}f(y_i)[1-F(y_i)]^{T_i}dy_i,& \qquad (16)
\end{aligned}
$$

where

$$T_k = S_k + \sum_{j_k=u_{(k)}-i_k+1}^{u_{(k+1)}-i_{k+1}} R_{j_k} + (i_k + u_{k+1} - i_{k+1}), \qquad k = 1, \cdots, l$$
$$T_{l+1} = S_{l+1}, \ \cdots \ , T_{m_2} = S_{m_2}.$$

We first note that the integrand in (16) corresponds to that of the joint density function of a progressively Type-II censored sample of size

$$N = m_2 + \sum_{i=1}^{m_2} T_i = n_1 + n_2 - \sum_{j=1}^{u_1-i_1} (R_j + 1),$$

where $R_1+R_2+\cdots+R_{m_1} = n_1-m_1$, and the progressive censoring scheme $(T_1, T_2, \cdots, T_{m_2})$. Therefore, the value of the integral in (16) is simply

$$\frac{1}{N(N-T_1-1)\cdots(N-T_1-\cdots-T_{m_2-1}-m_2+1)}.$$

As a result, we readily obtain from (16) the unconditional joint pmf of $\boldsymbol{U} = (U_1, \cdots, U_l)$ as

$$P(U_1 = u_1, \cdots, U_l = u_l) = AB \sum_{i_1=0}^{u_1} \ldots \sum_{i_l=0}^{u_l} \frac{\prod_{j=1}^{l+1} \gamma_{i_j,u_j}(R_{u_{(j-1)}+1}, \ldots, R_{u_{(j)}})}{\prod_{j=0}^{m_2-1}(N - T_{(j)} - j)}, \qquad (17)$$

where $T_{(0)} = 0$ and $T_{(j)} = T_1 + T_2 + \cdots + T_j$ for $j = 1, 2, \cdots$.

4 Special Case of Complete Samples and Precedence Statistic

In this section, we first show that in the special case of complete samples, the joint probability mass function of $(U_1, \cdots, U_l)$ reduces to the negative hypergeometric distribution. For this purpose, we need the following lemma.

Lemma 1 *For $l > 0$, we have*

$$\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} \frac{1}{l+i} = B(l, n+1),$$

where $B(a, b)$ denotes the complete beta function.

Proof: The result follows immediately by noticing that the beta integral

$$\begin{aligned} B(l, n+1) &= \int_0^1 t^{l-1}(1-t)^n dt = \int_0^1 \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} t^{i+l-1} dt \\ &= \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} \frac{1}{l+i}. \end{aligned}$$

We shall use this lemma now to establish that the joint pmf of $\boldsymbol{U} = (U_1, \cdots, U_l)$ reduces to the negative hypergeometric distribution in the case of complete samples. Let us set $R_1 = R_2 = \cdots = R_{m_1} = 0$ and $S_1 = S_2 = \cdots = S_{m_2} = 0$ so that $m_1 = n_1, m_2 = n_2, A = n_1!, B = n_2!$, and from (9)

$$\gamma_{i,l}(0, 0, \ldots, 0) = \frac{(-1)^i}{\left(\prod_{j=1}^{i} j \right) \left(\prod_{j=1}^{l-i} (l-i+1) \right)} = \frac{(-1)^i}{i!(l-i)!}.$$

Consequently, we have $T_j = i_j + u_{j+1} - i_{j+1}$ for $j = 1, 2, \cdots, l-1$,
$T_l = i_l + u_{l+1}$, $T_j = 0$ for $j = l+1, \cdots, m_2$, and $N = n_1 + n_2 - u_1 + i_1$.
So, the joint pmf of $(U_1, \cdots, U_l)$ in (17) reduces to

$$\begin{aligned} & P(U_1 = u_1, U_2 = u_2, \cdots, U_l = u_l) \\ &= n_1! n_2! \sum_{i_1=0}^{u_1} \cdots \sum_{i_l=0}^{u_l} \frac{\left(\prod_{j=1}^{l} \frac{(-1)^{i_j}}{i_j!(u_j-i_j)!} \right) \frac{1}{u_{l+1}!}}{\left(\prod_{k=1}^{l} (n_1 + n_2 - u_{(k)} + i_k - k + 1) \right)} \frac{1}{(n_2 - l)!} \\ &= \frac{n_1! n_2!}{(\prod_{k=1}^{l+1} u_k!)(n_2 - l)!} \prod_{k=1}^{l} \sum_{i_k=0}^{u_k} (-1)^{i_k} \begin{pmatrix} u_k \\ i_k \end{pmatrix} \frac{1}{n_1 + n_2 - u_{(k)} + i_k} \\ &= \frac{n_1! n_2!}{(\prod_{k=1}^{l+1} u_k!)(n_2 - l)!} \prod_{k=1}^{l} \frac{u_k!(n_1 + n_2 - u_{(k)} - k)!}{(n_1 + n_2 - u_{(l-1)} - l - 1)!}, \end{aligned}$$

where $u_{(0)} = 0$, with the last equality following by the use of Lemma 1. After noting that $u_{(l)} = n_1 - u_{l+1}$ and canceling some terms in the product, the above expression of the joint pmf reduces to

$$P(U_1 = u_1, \cdots, U_l = u_l) \quad = \quad \frac{n_1! n_2! (n_1 + n_2 - u_{(l)} - l)!}{(n_1 + n_2)!(n_2 - l)! u_{l+1}!} = \frac{\begin{pmatrix} n_2 + u_{l+1} - l \\ u_{l+1} \end{pmatrix}}{\begin{pmatrix} n_1 + n_2 \\ n_1 \end{pmatrix}} \ .$$

In particular, if we take $l = n_2$, the above joint pmf simply becomes the well-known result

$$P(U_1 = u_1, \cdots, U_{n_2} = u_{n_2}) = \frac{1}{\begin{pmatrix} n_1 + n_2 \\ n_1 \end{pmatrix}}.$$

Now, for the primary goal of testing the hypothesis that the two groups are the same against the hypothesis that the treatment group has larger lifetime than the control group, we propose the precedence test statistic $P_{(l)}$ which is simply the number of failures from the X-sample that precede the $l^t h$ failure from the Y-sample, i.e.

$$P_{(l)} = \sum_{i=1}^{l} U_i.$$

The null pmf of $P_{(l)}$ can be obtained easily from the joint pmf of the placements $(U_1, \cdots, U_l)$ presented in (17). Clearly, we will reject the null hypothesis $H_0 : F(x) = G(x)$ and accept the alternative hypothesis $H_1 : F(x) \geq G(x)$ for larger values of the precedence statistic $P_{(l)}$.

5 Numerical Results

In this section, we present the joint pmf of the placement statistics and the null distribution of the precedence statistic for several progressive Type-II censoring schemes and sample sizes from the two populations. From these tables, one can determine the critical values for a specified level of significance.

Table 1 provides the joint pmf of the placement statistics for $l = 3$ for the case when $n_1 = 15, n_2 = 15, m_1 = 5, m_2 = 5$, $(R_1, R_2, R_3, R_4, R_5) = (10, 0, 0, 0, 0)$ and

$(S_1, S_2, S_3, S_4, S_5) = (10, 0, 0, 0, 0)$. Table 2 provides the corresponding null pmf of the precedence statistic $P_{(3)}$. For a specified level of significance α, one can obtain the appropriate critical region by looking at the right-tail probabilities. In the interest of brevity, we will only provide the null pmf of $P_{(3)}$ for various progressive censoring schemes and sample sizes. Table 3 provides a list of combinations of n_1, n_2 with $m_1 = m_2 = 5$ and censoring schemes $\boldsymbol{R}$ and $\boldsymbol{S}$. In Table 4, we present the null distribution of the precedence statistic $P_{(3)}$ for all the cases listed in Table 3. Table 5 provides a list of combinations of unequal n_1, n_2 with $m_1 = m_2 = 5$ and censoring schemes $\boldsymbol{R}$ and $\boldsymbol{S}$. In Table 6, the null distribution of the precedence statistic is presented for all the cases listed in Table 5. Table 7 provides a list of censoring schemes $\boldsymbol{R}$ and $\boldsymbol{S}$ for the case $n_1 = n_2 = 20$ and $m_1 = m_2 = 10$. In Table 8, the null distribution of the precedence statistic $P_{(3)}$ is presented for all the cases listed in Table 7.

From these tables, we observe that when m_1 and m_2 are small, the support of the distribution of $P_{(l)}$ is small and its pmf is therefore concentrated over a small number of points extending from 0 to m_1. Consequently, there is a limited choice for the nominal level of significance and the corresponding critical region. On the other hand, when m_1 and m_2 are large, the support of the precedence statistic becomes wider and hence its pmf is distributed over a larger set of values. This means there are more choices for selecting nominal values of α. Furthermore, it is also clear from Tables 4, 6 and 8 that if more units are censored early in the experiment, the size of the critical region is usually larger as compared to the schemes when more units are censored later in the experiment, in which case the critical regions are of smaller size and are located farther into the right tail. The critical regions are also located farther into the right tail when the number of units censored are more evenly distributed over the number of observed failures.

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Table 1: Joint PMF of the Placement Statistics for $l = 3$

			$U_3 = 0$			
U_1/U_2	0	1	2	3	4	5
0	0.018	0.085	0.056	0.034	0.017	0.006
1	0.085	0.056	0.034	0.017	0.006	0.000
2	0.025	0.015	0.008	0.002	0.000	0.000
3	0.006	0.003	0.001	0.000	0.000	0.000
4	0.001	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
			$U_3 = 1$			
0	0.038	0.056	0.034	0.017	0.006	0.000
1	0.056	0.034	0.017	0.006	0.000	0.000
2	0.015	0.008	0.002	0.000	0.000	0.000
3	0.003	0.001	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
			$U_3 = 2$			
0	0.025	0.034	0.017	0.006	0.000	0.000
1	0.034	0.017	0.006	0.000	0.000	0.000
2	0.008	0.002	0.000	0.000	0.000	0.000
3	0.001	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
			$U_3 = 3$			
0	0.015	0.017	0.006	0.000	0.000	0.00
1	0.017	0.006	0.000	0.000	0.000	0.000
2	0.003	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
			$U_3 = 4$			
0	0.008	0.006	0.000	0.000	0.000	0.000
1	0.006	0.0000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000
			$U_3 = 5$			
0	0.003	0.000	0.000	0.000	0.000	0.000
1	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000	0.000

Table 2: Null Distribution of the Precedence Statistic $P_{(3)}$

$P_{(3)}$	Pmf	Survival Fn.
0	0.018	1.000
1	0.207	0.982
2	0.276	0.776
3	0.254	0.500
4	0.173	0.246
5	0.073	0.073

Table 3: Progressive Censoring Schemes: Equal Sample Size Case when $m_1 = m_2 = 5$

Scheme	n_1	n_2	$\boldsymbol{R}$					$\boldsymbol{S}$				
1	10	10	5	0	0	0	0	5	0	0	0	0
2	10	10	0	5	0	0	0	0	5	0	0	0
3	10	10	0	0	5	0	0	0	0	5	0	0
4	10	10	0	0	0	5	0	0	0	0	5	0
5	10	10	0	0	0	0	5	0	0	0	0	5
6	10	10	3	2	0	0	0	3	2	0	0	0
7	10	10	2	3	0	0	0	2	3	0	0	0
8	10	10	3	1	1	0	0	3	1	1	0	0
9	10	10	2	2	1	0	0	2	2	1	0	0
10	10	10	1	2	2	0	0	2	1	2	0	0
11	10	10	3	1	1	0	0	3	2	0	0	0
12	10	10	0	3	2	0	0	2	1	1	0	0
13	15	15	10	0	0	0	0	10	0	0	0	0
14	15	15	0	10	0	0	0	0	10	0	0	0
15	15	15	0	0	10	0	0	0	0	10	0	0
16	15	15	0	0	0	10	0	0	0	0	10	0
17	15	15	0	0	0	0	10	0	0	0	0	10
18	15	15	5	5	0	0	0	5	5	0	0	0
19	15	15	0	5	5	0	0	0	5	5	0	0
20	15	15	0	0	5	5	0	0	0	5	5	0
21	15	15	0	0	0	5	5	0	0	0	5	5
22	15	15	9	1	0	0	0	9	1	0	0	0
23	15	15	0	0	0	2	8	0	0	0	2	8
24	15	15	2	2	2	2	2	2	2	2	2	2

Table 4: Null Right Tail Probabilities of the Precedence Statistic $P_{(3)}$: Equal Sample Size Case

Scheme	0	1	2	3	4	5
1	1.000	0.967	0.767	0.500	0.249	0.075
2	1.000	0.945	0.834	0.500	0.223	0.059
3	1.000	0.895	0.709	0.500	0.154	0.024
4	1.000	0.895	0.709	0.500	0.314	0.052
5	1.000	0.895	0.709	0.500	0.314	0.175
6	1.000	0.957	0.804	0.500	0.233	0.064
7	1.000	0.952	0.817	0.500	0.228	0.062
8	1.000	0.946	0.768	0.500	0.210	0.050
9	1.000	0.941	0.781	0.500	0.204	0.047
10	1.000	0.931	0.776	0.519	0.197	0.041
11	1.000	0.957	0.804	0.559	0.263	0.073
12	1.000	0.931	0.795	0.535	0.204	0.042
13	1.000	0.982	0.776	0.500	0.246	0.073
14	1.000	0.960	0.879	0.500	0.212	0.054
15	1.000	0.888	0.701	0.500	0.109	0.012
16	1.000	0.888	0.701	0.500	0.326	0.036
17	1.000	0.888	0.701	0.500	0.326	0.195
18	1.000	0.969	0.852	0.500	0.218	0.057
19	1.000	0.916	0.765	0.500	0.130	0.017
20	1.000	0.888	0.701	0.500	0.255	0.028
21	1.000	0.888	0.701	0.500	0.326	0.143
22	1.000	0.979	0.799	0.500	0.235	0.066
23	1.000	0.888	0.701	0.500	0.326	0.178
24	1.000	0.917	0.738	0.500	0.262	0.083

Table 5: Progressive Censoring Schemes: Unequal Sample Size Case when $m_1 = m_2 = 5$

Scheme	n_1	n_2	$\boldsymbol{R}$					$\boldsymbol{S}$				
1	15	10	10	0	0	0	0	5	0	0	0	0
2	15	10	0	10	0	0	0	0	5	0	0	0
3	15	10	0	0	10	0	0	0	0	5	0	0
4	15	10	0	0	0	10	0	0	0	0	5	0
5	15	10	0	0	0	0	10	0	0	0	0	5
6	15	10	2	8	0	0	0	5	0	0	0	0
7	15	10	0	2	8	0	0	0	5	0	0	0
8	15	10	0	0	2	8	0	0	0	5	0	0
9	15	10	0	0	0	2	8	0	0	0	5	0
10	15	10	0	0	0	1	9	0	0	0	1	4
11	15	10	5	5	0	0	0	2	3	0	0	0
12	15	10	4	4	2	0	0	0	0	2	3	0
13	15	10	0	2	4	4	0	0	0	0	2	3
14	15	10	0	0	2	4	4	0	0	0	2	3
15	15	10	2	2	2	2	2	1	1	1	1	1

Table 6: Null Right Tail Probabilities of the Precedence Statistic $P_{(3)}$: Unequal Sample Size Case when $m_1 = m_2 = 5$

Scheme	0	1	2	3	4	5
1	1.000	0.989	0.796	0.526	0.264	0.080
2	1.000	0.975	0.918	0.571	0.260	0.070
3	1.000	0.948	0.841	0.699	0.229	0.038
4	1.000	0.948	0.841	0.699	0.545	0.096
5	1.000	0.948	0.841	0.699	0.545	0.399
6	1.000	0.986	0.943	0.658	0.343	0.106
7	1.000	0.975	0.918	0.816	0.390	0.107
8	1.000	0.948	0.841	0.699	0.518	0.091
9	1.000	0.948	0.841	0.699	0.545	0.374
10	1.000	0.948	0.841	0.699	0.545	0.387
11	1.000	0.979	0.888	0.559	0.260	0.071
12	1.000	0.948	0.789	0.472	0.145	0.023
13	1.000	0.948	0.841	0.673	0.413	0.070
14	1.000	0.948	0.841	0.699	0.518	0.277
15	1.000	0.960	0.856	0.680	0.446	0.193

Table 7: Progressive Censoring Schemes: Equal Sample Size Case when $m_1 = m_2 = 10$

Scheme	n_1	n_2	$\boldsymbol{R}$										$\boldsymbol{S}$									
1	20	20	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0	0	0
2	20	20	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0	0
3	20	20	0	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0
4	20	20	0	0	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0
5	20	20	0	0	0	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0
6	20	20	0	0	0	0	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0
7	20	20	0	0	0	0	0	0	0	0	1	9	0	0	0	0	0	0	0	0	1	9
8	20	20	5	5	0	0	0	0	0	0	0	0	5	5	0	0	0	0	0	0	0	0
9	20	20	0	0	0	0	0	0	0	0	5	5	0	0	0	0	0	0	0	0	5	5
10	20	20	2	2	3	3	0	0	0	0	0	0	2	2	3	3	0	0	0	0	0	0
11	20	20	0	0	0	0	0	0	2	2	3	3	0	0	0	0	0	0	2	2	3	3
12	20	20	0	0	0	0	0	0	3	3	2	2	0	0	0	0	0	0	3	3	2	2
13	20	20	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2	0	0	0	0	0
14	20	20	0	0	0	0	0	2	2	2	2	2	0	0	0	0	0	2	2	2	2	2
15	20	20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 8: Null Right Tail Probabilities of the Precedence Statistic $P_{(3)}$: Equal Sample Size Case when $m_1 = m_2 = 10$

Scheme	0	1	2	3	4	5	6	7	8	9	10
1	1.000	0.956	0.741	0.500	0.297	0.156	0.071	0.027	0.008	0.002	0.000
2	1.000	0.930	0.803	0.500	0.269	0.126	0.051	0.018	0.005	0.001	0.000
3	1.000	0.885	0.698	0.500	0.214	0.076	0.022	0.005	0.001	0.000	0.000
4	1.000	0.885	0.698	0.500	0.331	0.120	0.036	0.009	0.002	0.000	0.000
5	1.000	0.885	0.698	0.500	0.331	0.204	0.062	0.015	0.003	0.000	0.000
6	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.029	0.005	0.001	0.000
7	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.064	0.032	0.015	0.007
8	1.000	0.941	0.781	0.500	0.278	0.135	0.057	0.020	0.006	0.001	0.000
9	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.064	0.032	0.015	0.005
10	1.000	0.905	0.722	0.500	0.280	0.110	0.036	0.010	0.002	0.000	0.000
11	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.064	0.030	0.011	0.002
12	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.064	0.028	0.009	0.002
13	1.000	0.905	0.722	0.500	0.293	0.137	0.045	0.012	0.002	0.000	0.000
14	1.000	0.885	0.698	0.500	0.331	0.204	0.118	0.059	0.025	0.008	0.001
15	1.000	0.895	0.709	0.500	0.314	0.175	0.085	0.035	0.012	0.003	0.000