THE UNIVERSITY OF TEXAS AT SAN ANTONIO, COLLEGE OF BUSINESS

Working Paper SERIES

Date September 8, 2016

WP # 0003MSS-253-2016

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Arkadiusz Koziol
University of Zielona Gora

Anuradha Roy
University of Texas at San Antonio

Roman Zmyslony
University of Zielona Gora

Ricardo Leiva
F.C.D., Universidad Nacional de Cuyo

Miguel Fonseca
NOVA University of Lisbon

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Arkadiusz Kozioł
Computer Science and Econometrics
University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland
email: a.koziol@wmie.uz.zgora.pl

Anuradha Roy
Department of Management Science and Statistics
The University of Texas at San Antonio
San Antonio, TX 78249, USA
e-mail: anuradha.roy@utsa.edu

Roman Zmyślony
Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland
e-mail: r.zmyslony@wmie.uz.zgora.pl

Ricardo Leiva
Departamento de Matemática
F.C.E., Universidad Nacional de Cuyo, 5500 Mendoza, Argentina
email: rleiva@fcemail.uncu.edu.ar

Miguel Fonseca
Center of Mathematics and Applications, Faculty of Sciences and Technology
NOVA University of Lisbon
Monte da Caparica, 2829-516 Caparica, Portugal
email: fmig@fct.unl.pt

Abstract

The article addresses the best unbiased estimators of doubly exchangeable covariance structure for three-level data, an extension of the block compound symmetry covariance structure. Under multivariate normality, the free-coordinate approach is used to obtain linear and quadratic estimates for the model parameters that are sufficient, complete, unbiased and consistent. Data from a clinical study is analyzed to illustrate the application of the obtained results.

Keywords Best unbiased estimator, doubly exchangeable covariance structure, three-level multivariate data, coordinate free approach, unstructured mean vector.

2010 Mathematics Subject Classification 62H12, 62J10, 62F10

1 Introduction

Multi-level multivariate observations are becoming increasingly visible across all fields of biomedical, medical and engineering among many others these days. This article deals with the estimation and best unbiased estimators of doubly exchangeable covariance structure (defined in Section 2) for three-level multivariate observations (m dimensional observation vector repeatedly measured at u locations and over v time points). Consider an example from a clinical trial of glaucoma. Glaucoma is a group of eye diseases that lead to the damage of the optic nerve. Over the years, numerous investigators have studied the characteristics of individuals who have glaucoma. Those studies identified several factors such as intraocular pressure (IOP), and central corneal thickness (CCT), useful in the diagnosis of glaucoma. Measurements of intraocular pressure (IOP) and central corneal thickness (CCT) are obtained from both the eyes (sites), each at three time points at an interval of three months for 30 patients. It is clear that for this data set $m = 2$, $u = 2$ and $v = 3$. This example will be used later in Section 5 for an illustrative purpose. Our main intention of the analysis of this data set is to illustrate the proposed methods rather than giving any insight into the data set itself.

For this data set the unstructured variance-covariance matrix is $(12 \times 12)-$dimensional, and therefore the number of unknown parameters in the unstructured variance-covariance matrix is 78. As a result, estimation of this unstructured variance-covariance matrix is not possible for small sample situations. Therefore, an assumption of doubly exchangeable (DE) covariance structure is necessary for small sample situation. DE covariance structure provides a substantial reduction in the number of unknown covariance parameters to just 9, and thus, may help in providing the correct information about the true association of the three-level multivariate data with small samples. Roy and Leiva (2007) and Leiva and Roy (2011, 2012) used this data set in their studies, and assumed that the data have DE covariance structure. The structure is simply implied by the organization (design in broad sense) of the experiment, and need not be tested all the times.

For three-level multivariate observations both u and v must be greater than 1; i.e., both $u > 1$ and $v > 1$. If either $u = 1$ or $v = 1$, the data become two-level or doubly multivariate with blocked compound symmetry (BCS) covariance structure, and finally if both $u = 1$ and $v = 1$, the data just become classical multivariate data with unstructured variance-covariance matrix. If $m = 1$ with either $u = 1$ or $v = 1$, the data also become classical multivariate data, but with compound symmetry covariance structure.

Doubly exchangeable covariance structure was first studied by Roy and Leiva (2007) in the context of classification rules for three-level multivariate data. Later these two authors wrote a series of articles on classification rules for three-level multivariate data with different covariance structures and with different mean vectors: among them Leiva and Roy (2009, 2011, 2012) are worth mentioning. Coelho and Roy (2014) studied hypothesis testing problem of this DE covariance structure. Roy and Fonseca (2012) studied this DE covariance structure while developing general linear model for three-level multivariate data with error vectors having DE covariance structure. They derived unbiased estimators of the component matrices of the orthogonally transformed DE covariance structure for testing the intercept and slope parameters of the general linear model using parametric bootstrap as well as multivariate Satterthwaite approximation. Recently, Roy (2014) proposed a two-stage principal component analysis of interval data using BCS and DE covariance structures. To derive the principal components of the interval data Roy (2014) obtained the unbiased estimates of BCS and DE covariance structures by considering the interval data as two-level and three-level multivariate data respectively. One might ask at this point whether these unbiased estimates are reasonably good or not. Optimal estimation for two-level multivariate data with unstructured mean vector (fixed parameters) and with BCS covariance structure set-up was studied by Roy et al. (2016). Recently, Kozioł et al. (2015) studied the same problem with structured mean vector, where mean vector remains constant over sites, i.e., $\mathbf{1}_u \otimes \boldsymbol{\mu}$ with $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$. In this article, optimal unbiased estimators for three-level multivariate data with unstructured mean vector and with DE covariance structure will be constructed.

The assumption of double exchangeability reduces the number of unknown parameters considerably, thus allows more dependable or reliable parameter estimates. This covariance structure can capture the data arrangement or data pattern in a three-level multivariate data, and thus may offer more information about the true association of the data. One of the many advantages of this covariance structure is that the repeated measurements at any level need not be of equally spaced. The unstructured variance-covariance matrix has $vum(vum+1)/2$ unknown parameters, which can be large for arbitrary values of v, u or m. In order to reduce the number of unknown parameters, it is then essential to assume some appropriate structure on the variance-covariance matrix. One may assume a DE covariance structure in this situation, where the data is multivariate in three levels. DE covariance structure has only $3m(m+1)/2$ unknown parameters. This number does not even depend on the number of locations or sites u and the number of time points v. The use of DE covariance structure provides a better insight into the three-level data structure. The problem of interest in this paper is to find optimal estimators of DE covariance structure. To the best of the authors' knowledge, none of the previous studies have considered the estimation properties of the

DE covariance structure.

This article derives the unbiased estimators of the unstructured mean vector and the DE covariance structure, and addresses the issue of optimal properties of these unbiased estimators that is motivated by the example of the clinical trial of glaucoma that is discussed earlier in the Introduction. A characterization of best linear unbiased estimator (BLUE) given by Zmyślony (1978) and completeness in Zmyślony (1980) are used to derive the optimal properties of unbiased estimators of the DE covariance matrix. The derivation and computation of these estimators are developed using the coordinate free approach $\big($see Kruskal (1968) and Drygas (1970)$\big)$.

2 Doubly exchangeable covariance structure

The $(vum \times vum)-$dimensional DE covariance structure is defined as

$$\begin{aligned}\mathbf{\Gamma} &= \begin{bmatrix} \mathbf{\Sigma}_0 & \mathbf{\Sigma}_1 & \ldots & \mathbf{\Sigma}_1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{\Sigma}_1 & \mathbf{\Sigma}_1 & \ldots & \mathbf{\Sigma}_0 \end{bmatrix} \\ &= \boldsymbol{I}_v \otimes (\mathbf{\Sigma}_0 - \mathbf{\Sigma}_1) + \boldsymbol{J}_v \otimes \mathbf{\Sigma}_1, \end{aligned} \tag{2.1}$$

where $\boldsymbol{I}_v$ is the $v \times v$ identity matrix, $\mathbf{1}_v$ is a $v \times 1$ vector of ones, $\boldsymbol{J}_v = \mathbf{1}_v \mathbf{1}_v'$, $\otimes$ represents the Kronecker product and

$$\begin{aligned}\mathbf{\Sigma}_0 &= \boldsymbol{I}_u \otimes (\mathbf{\Gamma}_0 - \mathbf{\Gamma}_1) + \boldsymbol{J}_u \otimes \mathbf{\Gamma}_1, \quad \text{and} \\ \mathbf{\Sigma}_1 &= \boldsymbol{J}_u \otimes \mathbf{\Gamma}_2.\end{aligned}$$

We assume $\mathbf{\Sigma}_0$ is a positive definite symmetric $um \times um$ matrix, and $\mathbf{\Sigma}_1$ is a symmetric $um \times um$ matrix, and $\mathbf{\Sigma}_0 - \mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_0 + (v-1)\mathbf{\Sigma}_1$ are positive definite matrices, so that the $vum \times vum$ matrix $\mathbf{\Gamma}$ is positive definite $\big($for a proof, see Lemma 2.1 in Roy and Leiva (2011)$\big)$.

We see that the matrix $\mathbf{\Gamma}$ is exchangeable with matrix parameters $\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$, and the matrix $\mathbf{\Sigma}_0$ is exchangeable with the matrix parameters $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$. Because of this doubly exchangeable nature of this covariance structure $\mathbf{\Gamma}$, it is called doubly exchangeable covariance structure, and can equivalently be written as follows

$$\mathbf{\Gamma} = \boldsymbol{I}_v \otimes \mathbf{\Sigma}_0 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \mathbf{\Sigma}_1, \tag{2.2}$$

We can write this doubly exchangeable covariance structure $\mathbf{\Gamma}$ in terms of $\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ as

$$\mathbf{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes (\mathbf{\Gamma}_0 - \mathbf{\Gamma}_1) + \boldsymbol{I}_v \otimes \boldsymbol{J}_u \otimes (\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2) + \boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \mathbf{\Gamma}_2,$$

which can equivalently be written as

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2. \tag{2.3}$$

This last form (2.3) will be used to build orthogonal basis with respect to trace of inner product base for components of matrix $\boldsymbol{\Gamma}$.

We assume $\boldsymbol{\Gamma}_0$ is a positive definite symmetric $m \times m$ matrix, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are a symmetric $m \times m$ matrices, and $\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1 - u\boldsymbol{\Gamma}_2$, $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1 + (v-1)u\boldsymbol{\Gamma}_2$ are positive definite matrices, so that the $vum \times vum$ matrix $\boldsymbol{\Gamma}$ is positive definite $\big($for a proof, see Lemma 3.1 in Roy and Fonseca (2012)$\big)$. The $m \times m$ block diagonals $\boldsymbol{\Gamma}_0$ in $\boldsymbol{\Gamma}$ represent the variance-covariance matrix of the m response variables at any given time point and at any given site, whereas the $m \times m$ block off diagonals $\boldsymbol{\Gamma}_1$ in $\boldsymbol{\Gamma}$ represent the covariance matrix of the m response variables at any given time point and between any two sites. The $m \times m$ block off diagonals $\boldsymbol{\Gamma}_2$ in $\boldsymbol{\Gamma}$ represent the covariance matrix of the m response variables between any two time points. We assume $\boldsymbol{\Gamma}_0$ is constant for all time points and sites, $\boldsymbol{\Gamma}_1$ is same between any two sites and for all time points and $\boldsymbol{\Gamma}_2$ is assumed to be the same for any pair of time points, irrespective of the same site or between any two sites. We derive the unbiased estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ in the following section.

3 Unbiased estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$

Let $\boldsymbol{y}_{r,ts}$ be a m-variate vector of measurements on the rth individual at the tth time point and at the sth site; $r = 1, \ldots, n$, $t = 1, \ldots, v$, $s = 1, \ldots, u$. The n individuals are all independent. Let $\boldsymbol{y}_r = (\boldsymbol{y}'_{r,11}, \ldots, \boldsymbol{y}'_{r,vu})'$ be the vum-variate vector of all measurements corresponding to the rth individual. Finally, let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ be a random sample of size n drawn from the population $N_{vum}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$, where $\boldsymbol{\mu} \in \mathbb{R}^{vum}$ with $\boldsymbol{\mu} = (\boldsymbol{\mu}'_{11}, \ldots, \boldsymbol{\mu}'_{vu})'$ and $\boldsymbol{\Gamma}$ is assumed to be a $vum \times vum$ positive definite matrix.

Theorem 1. *Under this set up unbiased estimators of $\boldsymbol{\mu}$, $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are respectively*

$$\widetilde{\boldsymbol{\mu}} = \frac{1}{n}\sum_{r=1}^{n} \boldsymbol{y}_r, \tag{3.4}$$

$$\widetilde{\boldsymbol{\Gamma}}_0 = \frac{1}{(n-1)vu}\boldsymbol{C}_0 = \frac{1}{(n-1)vu}\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$

$$\widetilde{\boldsymbol{\Gamma}}_1 = \frac{1}{(n-1)vu\,(u-1)}\boldsymbol{C}_1 = \frac{1}{(n-1)vu\,(u-1)}\sum_{t=1}^{v}\sum_{s=1}^{u}\underset{s \neq s^*}{\sum_{s^*=1}^{u}}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts^*} - \overline{\boldsymbol{y}}_{\bullet,ts^*}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$

and

$$\widetilde{\mathbf{\Gamma}}_2 = \frac{1}{(n-1)v\,(v-1)\,u^2}\boldsymbol{C}_2 = \frac{1}{(n-1)v\,(v-1)\,u^2}\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{t\neq t^*=1}^{v}\sum_{s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,t^*s^*}-\overline{\boldsymbol{y}}_{\bullet,t^*s^*}\right)\left(\boldsymbol{y}_{r,ts}-\overline{\boldsymbol{y}}_{\bullet,ts}\right)',$$

where $\overline{\boldsymbol{y}}_{\bullet,ts} = \frac{1}{n}\sum_{r=1}^{n}\boldsymbol{y}_{r,ts}$, *for* $t = 1,\ldots,v$ *and* $s = 1,\ldots,u$.

***Proof*:** Clearly, $\overline{\boldsymbol{y}} = (\overline{\boldsymbol{y}}'_{\bullet,11},\ldots,\overline{\boldsymbol{y}}'_{\bullet,vu})' \sim N_{vum}\left(\boldsymbol{\mu},\frac{1}{n}\mathbf{\Gamma}\right)$ with $\boldsymbol{\mu} = (\boldsymbol{\mu}'_{11},\ldots,\boldsymbol{\mu}'_{vu})'$ and $\mathbf{\Gamma}$ is defined in (2.1). The independence of $\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_n$ and the doubly exchangeability of $\mathbf{\Gamma}$ give

$$\operatorname{cov}\left[\boldsymbol{y}_{r,ts};\boldsymbol{y}_{r,t^*s^*}\right] = \mathrm{E}\left[\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)\left(\boldsymbol{y}_{r,t^*s^*}-\boldsymbol{\mu}_{t^*s^*}\right)'\right] = \begin{cases}\mathbf{\Gamma}_0 & \text{if} \quad t=t^*, s=s^*\\ \mathbf{\Gamma}_1 & \text{if} \quad t=t^*, s\neq s^*\\ \mathbf{\Gamma}_2 & \text{if} \quad t\neq t^*,\end{cases}$$

and

$$\operatorname{cov}\left[\overline{\boldsymbol{y}}_{\bullet,ts};\overline{\boldsymbol{y}}_{\bullet,t^*s^*}\right] = \mathrm{E}\left[\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)\left(\overline{\boldsymbol{y}}_{\bullet,t^*s^*}-\boldsymbol{\mu}_{t^*s^*}\right)'\right] = \begin{cases}\frac{1}{n}\mathbf{\Gamma}_0 & \text{if} \quad t=t^*, s=s^*\\ \frac{1}{n}\mathbf{\Gamma}_1 & \text{if} \quad t=t^*, s\neq s^*\\ \frac{1}{n}\mathbf{\Gamma}_2 & \text{if} \quad t\neq t^*.\end{cases}$$

Let

$$\begin{aligned}\boldsymbol{C}_0 &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts}-\overline{\boldsymbol{y}}_{\bullet,ts}\right)\left(\boldsymbol{y}_{r,ts}-\overline{\boldsymbol{y}}_{\bullet,ts}\right)' \qquad (3.5)\\ &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left[\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)-\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)\right]\left[\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)-\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)\right]'\\ &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)' - n\sum_{t=1}^{v}\sum_{s=1}^{u}\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)',\end{aligned}$$

then

$$\begin{aligned}E\left[\boldsymbol{C}_0\right] &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{r=1}^{n}E\left[\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)'\right]\\ &\quad -n\sum_{t=1}^{v}\sum_{s=1}^{u}E\left[\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)'\right]\\ &= nvu\mathbf{\Gamma}_0 - vu\mathbf{\Gamma}_0 = (n-1)\,vu\mathbf{\Gamma}_0.\end{aligned}$$

Similarly, let

$$\begin{aligned}\boldsymbol{C}_1 &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts^*}-\overline{\boldsymbol{y}}_{\bullet,ts^*}\right)\left(\boldsymbol{y}_{r,ts}-\overline{\boldsymbol{y}}_{\bullet,ts}\right)' \qquad (3.6)\\ &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,ts^*}-\boldsymbol{\mu}_{ts^*}\right)\left(\boldsymbol{y}_{r,ts}-\boldsymbol{\mu}_{ts}\right)'\\ &\quad n\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u}\left(\overline{\boldsymbol{y}}_{\bullet,ts^*}-\boldsymbol{\mu}_{ts^*}\right)\left(\overline{\boldsymbol{y}}_{\bullet,ts}-\boldsymbol{\mu}_{ts}\right)',\end{aligned}$$

and so

$$E\left[\boldsymbol{C}_1\right] = (n-1)\, vu\, (u-1)\, \boldsymbol{\Gamma}_1.$$

Finally, let

$$\begin{aligned}\boldsymbol{C}_2 &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{t\neq t^*=1}^{v}\sum_{s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,t^*s^*} - \overline{\boldsymbol{y}}_{\bullet,t^*s^*}\right)\left(\boldsymbol{y}_{r,ts} - \overline{\boldsymbol{y}}_{\bullet,ts}\right)' \\ &= \sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{t\neq t^*=1}^{v}\sum_{s^*=1}^{u}\sum_{r=1}^{n}\left(\boldsymbol{y}_{r,t^*s^*} - \boldsymbol{\mu}_{t^*s^*}\right)\left(\boldsymbol{y}_{r,ts} - \boldsymbol{\mu}_{ts}\right)' \\ &\quad n\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{t\neq t^*=1}^{v}\sum_{s^*=1}^{u}\left(\overline{\boldsymbol{y}}_{\bullet,t^*s^*} - \boldsymbol{\mu}_{t^*s^*}\right)\left(\overline{\boldsymbol{y}}_{\bullet,ts} - \boldsymbol{\mu}_{ts}\right)'\end{aligned} \tag{3.7}$$

and so

$$E\left[\boldsymbol{C}_2\right] = (n-1)\, v\, (v-1)\, u^2 \boldsymbol{\Gamma}_2.$$

Therefore, unbiased estimators of $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are

$$\widetilde{\boldsymbol{\Gamma}}_0 = \frac{1}{(n-1)\, vu}\boldsymbol{C}_0, \tag{3.8}$$

$$\widetilde{\boldsymbol{\Gamma}}_1 = \frac{1}{(n-1)\, vu\, (u-1)}\boldsymbol{C}_1 \tag{3.9}$$

and

$$\widetilde{\boldsymbol{\Gamma}}_2 = \frac{1}{(n-1)\, v\, (v-1)\, u^2}\boldsymbol{C}_2, \tag{3.10}$$

where $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are given by (3.5), (3.6) and (3.7), respectively. Consequently,

$$\widetilde{\boldsymbol{\Gamma}} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes (\widetilde{\boldsymbol{\Gamma}}_0 - \widetilde{\boldsymbol{\Gamma}}_1) + \boldsymbol{I}_v \otimes \boldsymbol{J}_u \otimes (\widetilde{\boldsymbol{\Gamma}}_1 - \widetilde{\boldsymbol{\Gamma}}_2) + \boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \widetilde{\boldsymbol{\Gamma}}_2$$

is an unbiased estimator of $\boldsymbol{\Gamma}$. The optimal properties of the unbiased estimators are discussed in the following section.

4 Best unbiased estimators

In this section, optimal properties of unbiased estimators for mean vector and the covariance matrix $\boldsymbol{\Gamma}$ are obtained. Let the data matrix be $\underset{vum\times n}{\boldsymbol{Y}'} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n)$. Thus, the following column vector

$$\underset{nvum\times 1}{\boldsymbol{y}} = vec(\underset{vum\times n}{\boldsymbol{Y}'}) \sim N\big((\boldsymbol{1}_n \otimes \boldsymbol{I}_{vum})\boldsymbol{\mu}, \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}_{vum}\big).$$

This means that n independent random column vectors are identically distributed with $(vum \times vum)-$dimensional variance-covariance matrix

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2.$$

Define the projection matrix $\boldsymbol{P}$ as follows:

$$\boldsymbol{P} = \frac{1}{n}\boldsymbol{J}_n \otimes \boldsymbol{I}_{vum}, \tag{4.11}$$

and $\boldsymbol{V} = \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}_{vum}$ is the covariance matrix of $\boldsymbol{y}$. It is clear that $\boldsymbol{P}$ is an orthogonal projector on the subspace of the mean vector of $\boldsymbol{y}$. If $\boldsymbol{I}_n \otimes \boldsymbol{I}_{vum} \in \boldsymbol{\vartheta} = sp\{\boldsymbol{V}\}$, from (Gnot et al., 1980) it follows that $\boldsymbol{Py}$ is the best linear unbiased estimator (BLUE) if and only if $\boldsymbol{P}$ commutes with all covariance matrices $\boldsymbol{V}$. Therefore, we have the following results.

Result 1. *The projection matrix $\boldsymbol{P}$ commutes with the covariance matrix $\boldsymbol{V}$, i.e., $\boldsymbol{PV} = \boldsymbol{VP}$, where $\boldsymbol{V} = \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}$, the covariance matrix of $\boldsymbol{y}$.*

Proof. Now,

$$\begin{aligned} \boldsymbol{P} &= (\boldsymbol{1}_n \otimes \boldsymbol{I}_{vum})(\boldsymbol{1}_n \otimes \boldsymbol{I}_{vum})^+ \\ &= (\boldsymbol{1}_n \otimes \boldsymbol{I}_{vum})\left(\frac{1}{n}\boldsymbol{1}_n' \otimes \boldsymbol{I}_{vum}\right) \\ &= \frac{1}{n}\boldsymbol{J}_n \otimes \boldsymbol{I}_{vum}. \end{aligned}$$

Note that $(\frac{1}{n}\boldsymbol{J}_n \otimes \boldsymbol{I}_{vum})(\boldsymbol{I}_n \otimes \boldsymbol{\Gamma}) = \frac{1}{n}\boldsymbol{J}_n \otimes \boldsymbol{\Gamma}$ is symmetric. It implies that the matrix $\boldsymbol{P}$ commutes with the covariance matrix of $\boldsymbol{y}$. □

Lemma 1. *Let $\boldsymbol{\vartheta}$ denote the subspace spanned by $\boldsymbol{V}$, i.e., $\boldsymbol{\vartheta} = sp\{\boldsymbol{V}\}$. Then, $\boldsymbol{\vartheta}$ is a quadratic subspace. That is, ϑ is a linear space and if $\boldsymbol{V} \in \vartheta$ then $\boldsymbol{V}^2 \in \vartheta$ (see Seely (1971) for the definition).*

Proof. It is sufficient to prove that $\boldsymbol{\Gamma}^2 \in sp\{\boldsymbol{\Gamma}\}$. From the structure of the covariance matrix $\boldsymbol{V}$ it is clear that $sp\{\boldsymbol{V}\}$ is a quadratic subspace if and only if $sp\{\boldsymbol{\Gamma}\}$ is a quadratic subspace. Using the definition (2.1) of $\boldsymbol{\Gamma}$ after simple algebraic calculations one can find that

$$\boldsymbol{\Gamma}^2 = \boldsymbol{I}_v \otimes (\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_1)^2 + \boldsymbol{J}_v \otimes [\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_0\boldsymbol{\Sigma}_1 + (v-2)\boldsymbol{\Sigma}_1^2]. \tag{4.12}$$

By defining the component matrices $\boldsymbol{\Sigma}_0^* = \boldsymbol{\Sigma}_0^2 + (v-1)\boldsymbol{\Sigma}_1^2$ and $\boldsymbol{\Sigma}_1^* = \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_0\boldsymbol{\Sigma}_1 + (v-2)\boldsymbol{\Sigma}_1^2$, $\boldsymbol{\Gamma}^2$ in (4.12) can be rewritten as

$$\boldsymbol{\Gamma}^2 = \boldsymbol{I}_v \otimes (\boldsymbol{\Sigma}_0^* - \boldsymbol{\Sigma}_1^*) + \boldsymbol{J}_v \otimes \boldsymbol{\Sigma}_1^*.$$

This proves that $sp\{\boldsymbol{\Gamma}\} = sp\{\boldsymbol{\Gamma}^2\}$, and it implies $sp\{\boldsymbol{V}\}$ is a quadratic subspace. □

Now, because orthogonal projector on the space generated by the mean vector commutes with all covariances matrices, there exists BLUE for each estimable function of mean. Moreover, BLUE are least squares estimators (LSE), in view of Result 1. Thus, $\widetilde{\boldsymbol{\mu}}$ is the unique solution of the following normal equation

$$(\mathbf{1}_n \otimes \boldsymbol{I}_{vum})'(\mathbf{1}_n \otimes \boldsymbol{I}_{vum})\boldsymbol{\mu} = (\mathbf{1}_n \otimes \boldsymbol{I}_{vum})'\boldsymbol{y} \quad \text{or}$$
$$n\boldsymbol{I}_{vum}\boldsymbol{\mu} = [\boldsymbol{I}_{vum}, \boldsymbol{I}_{vum}, \ldots, \boldsymbol{I}_{vum}]\boldsymbol{y},$$

which means that

$$\widetilde{\boldsymbol{\mu}} = \frac{1}{n}\sum_{r=1}^{n} \boldsymbol{y}_r.$$

Let $\boldsymbol{M} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{I}_m - \boldsymbol{P}$. So, $\boldsymbol{M}$ is idempotent. Now, since $\boldsymbol{PV} = \boldsymbol{VP}$, and $\boldsymbol{\vartheta}$ is a quadratic space, $\boldsymbol{M\vartheta M} = \boldsymbol{M\vartheta}$ is also a quadratic space. We now construct a base for this quadratic subspace $\boldsymbol{\vartheta}$. We define

$$\boldsymbol{A}_{ii} = \boldsymbol{E}_{ii} \quad \text{and} \quad \boldsymbol{A}_{ij} = \boldsymbol{E}_{ij} + \boldsymbol{E}_{ji}, \quad \text{for } i < j; \text{ and } \ j = 1, \ldots, m,$$

as a base for symmetric matrices $\boldsymbol{\Gamma}$. The $(m \times m)-$dimensional matrices $\boldsymbol{E}_{ij}$ has 1 only at the ijth element, and 0 at all other elements. Then it is clear that the base for diagonal matrices of the form $\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(0)} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \le j, \quad j = 1, \ldots, m, \tag{4.13}$$

the base for matrices of the form $\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(1)} = \boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \le j, \quad j = 1, \ldots, m$$

and the base for matrices of the form $\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2$ is constituted by matrices

$$\boldsymbol{K}_{ij}^{(2)} = \boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij}, \quad \text{for } i \le j, \quad j = 1, \ldots, m.$$

It is clear from (2.2) that above base is orthogonal with respect to trace of inner product.

Result 2. *The complete and minimal sufficient statistics for the mean vector and the variance-covariance matrix are*

$$(\mathbf{1}_n' \otimes \boldsymbol{I}_{vum})\boldsymbol{y} \tag{4.14}$$
$$\textit{and} \qquad \boldsymbol{y}'\boldsymbol{M}\boldsymbol{K}_{ij}^{(l)}\boldsymbol{M}\boldsymbol{y},\ l = 0, 1, 2, \tag{4.15}$$

where $\boldsymbol{M} = \boldsymbol{I}_{nvum} - \boldsymbol{P}$ *and* $\boldsymbol{P}$ *is given in (4.11), see Fonseca et al. (2010), Seely (1977) and Zmyślony (1980).*

Now we prove that $\widetilde{\boldsymbol{\Gamma}}_{vum}$ is the best quadratic unbiased estimator (BQUE) for $\boldsymbol{\Gamma}$. Since $\boldsymbol{P}$ commutes with the covariance matrix of $\boldsymbol{y}$, for each parameter of covariance there exists BQUE if and only if

$$sp\{\boldsymbol{MVM}\}, \quad \text{where} \quad \boldsymbol{M} = \boldsymbol{I}_{nvum} - \boldsymbol{P},$$

is a quadratic subspace (see Zmyślony (1976) and Gnot et al. (1976, 1977)) or Jordan algebra (see Jordan et al. (1934)), where $\boldsymbol{V}$ stands for covariance matrix of $\boldsymbol{y}$. It is clear that if $sp\{\boldsymbol{V}\}$ is a quadratic subspace and if for each $\boldsymbol{\Sigma} \in sp\{\boldsymbol{V}\}$ commutativity $\boldsymbol{P\Sigma} = \boldsymbol{\Sigma P}$ holds, then $sp\{\boldsymbol{MVM}\} = sp\{\boldsymbol{MV}\}$ is also a quadratic subspace. According to the coordinate free approach, the expectation of $\boldsymbol{Myy'M}$ can be written as a linear combination of vectors $\text{vec}(\boldsymbol{MK}_{ij}^{(0)})$, $\text{vec}(\boldsymbol{MK}_{ij}^{(1)})$ and $\text{vec}(\boldsymbol{MK}_{ij}^{(2)})$ with unknown coefficients $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, respectively. Note also that identity covariance operator of $\boldsymbol{yy'}$ belongs to $sp\{\text{cov}(\boldsymbol{yy'})\}$. It implies that the ordinary best quadratic estimators are least square estimators for corresponding parameters $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, and they are calculated independently. Defining $\frac{m(m+1)}{2}$ column vectors $\boldsymbol{\sigma}^{(l)} = [\sigma_{ij}^{(l)}]$ for $i \leq j = 1, \ldots, m;\ l = 0, 1, 2$, we see that the normal equations have the following block diagonal structure

$$\left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}} \right) \begin{bmatrix} \boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)} \end{bmatrix}, \tag{4.16}$$

where for $i \leq j = 1, \ldots, m$; $a = \text{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(0)})^2\big)$, $b = \text{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(1)})^2\big)$ and $c = \text{tr}\big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(2)})^2\big)$, while the $\frac{m(m+1)}{2} \times 1$ vector $\boldsymbol{r}^{(l)} = \frac{1}{2-\boldsymbol{\delta}_{ij}}\big[r'\boldsymbol{K}_{ij}^{(l)}r\big]$ for $l = 0, 1, 2$, $\boldsymbol{\delta}_{ij}$ is the Kronecker delta and $\boldsymbol{r}$ stands for the residual vector, i.e., $\boldsymbol{r} = \boldsymbol{My} = (\boldsymbol{I}_{nvum} - \boldsymbol{P})\boldsymbol{y}$. Now to prove (4.16), we consider the following six cases:

Case 1: for $l = 0, i = j$ we have

$$\begin{aligned} \text{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(0)})^2\Big) &= \text{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ii}^2\Big] \\ &= \text{tr}(\boldsymbol{A}_{ii}^2)\text{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\text{tr}(\boldsymbol{I}_v)\text{tr}(\boldsymbol{I}_u) \\ &= (n-1)vu \\ \text{and} \quad (\boldsymbol{My})'\boldsymbol{K}_{ii}^{(0)}\boldsymbol{My} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ii})\boldsymbol{r} \\ &= \sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u} r_{rtsi}^2. \end{aligned}$$

Case 2: for $l = 0, i < j$ we have

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(0)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ij}^2\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\mathrm{tr}(\boldsymbol{I}_v)\mathrm{tr}(\boldsymbol{I}_u) \\
&= 2(n-1)vu \\
\text{and} \quad (\boldsymbol{M}\boldsymbol{y})'\boldsymbol{K}_{ij}^{(0)}\boldsymbol{M}\boldsymbol{y} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \boldsymbol{A}_{ij})\boldsymbol{r} \\
&= 2\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u} r_{rtsi}r_{rtsj}, \quad \text{for } i < j.
\end{aligned}$$

Case 3: for $l = 1, i = j$, one gets the estimators by noting the following

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(1)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u)^2 \otimes \boldsymbol{A}_{ii}^2\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ii}^2)\mathrm{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\mathrm{tr}(\boldsymbol{I}_v)\mathrm{tr}((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \\
&= (n-1)vu(u-1) \\
\text{and} \quad (\boldsymbol{M}\boldsymbol{y})'\boldsymbol{K}_{ii}^{(1)}\boldsymbol{M}\boldsymbol{y} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ii})\boldsymbol{r} \\
&= \sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u} r_{rtsi}r_{rts^*i}.
\end{aligned}$$

Case 4: for $l = 1, i < j$

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(1)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u)^2 \otimes \boldsymbol{A}_{ij}^2\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\mathrm{tr}(\boldsymbol{I}_v)\mathrm{tr}((u-2)\boldsymbol{J}_u + \boldsymbol{I}_u) \\
&= 2(n-1)vu(u-1) \\
\text{and} \quad (\boldsymbol{M}\boldsymbol{y})'\boldsymbol{K}_{ij}^{(1)}\boldsymbol{M}\boldsymbol{y} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{A}_{ij})\boldsymbol{r} \\
&= 2\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u} r_{rtsi}r_{rts^*j}, \quad \text{for } \ i < j.
\end{aligned}$$

Case 5: for $i = j$ we have

$$\begin{aligned}
\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ii}^{(2)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v)^2 \otimes \boldsymbol{J}_u^2 \otimes \boldsymbol{A}_{ii}^2\Big] \\
&= \mathrm{tr}(\boldsymbol{A}_{ii}^2)\mathrm{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\mathrm{tr}((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v)\mathrm{tr}(u\boldsymbol{J}_u) \\
&= (n-1)v(v-1)u^2 \\
\text{and} \quad (\boldsymbol{M}\boldsymbol{y})'\boldsymbol{K}_{ii}^{(2)}\boldsymbol{M}\boldsymbol{y} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ii})\boldsymbol{r} \\
&= \sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{t\neq t^*=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u} r_{rtsi}r_{rt^*s^*i}.
\end{aligned}$$

Case 6: for $i < j$ we have

$$\begin{aligned}\mathrm{tr}\Big(\boldsymbol{M}(\boldsymbol{K}_{ij}^{(2)})^2\Big) &= \mathrm{tr}\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big) \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v)^2 \otimes \boldsymbol{J}_u^2 \otimes \boldsymbol{A}_{ij}^2\Big] \\ &= \mathrm{tr}(\boldsymbol{A}_{ij}^2)\mathrm{tr}\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\mathrm{tr}((v-2)\boldsymbol{J}_v + \boldsymbol{I}_v)\mathrm{tr}(u\boldsymbol{J}_u) \\ &= 2(n-1)v(v-1)u^2 \\ \text{and} \quad (\boldsymbol{My})'\boldsymbol{K}_{ij}^{(2)}\boldsymbol{My} &= \boldsymbol{r}'(\boldsymbol{I}_n \otimes (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \boldsymbol{A}_{ij})\boldsymbol{r} \\ &= 2\sum_{r=1}^{n}\sum_{t=1}^{v}\sum_{t\neq t^*=1}^{v}\sum_{s=1}^{u}\sum_{s\neq s^*=1}^{u} r_{rtsi}r_{rt^*s^*j}, \quad \text{for} \quad i<j.\end{aligned}$$

Thus, to find the best quadratic unbiased estimator $\widetilde{\boldsymbol{\Gamma}}$ for $\boldsymbol{\Gamma}$, the following normal equation has to be solved

$$\left(\begin{bmatrix}(n-1)vu & 0 & 0 \\ 0 & (n-1)vu(u-1) & 0 \\ 0 & 0 & (n-1)v(v-1)u^2\end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}}\right)\begin{bmatrix}\boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)}\end{bmatrix} = \begin{bmatrix}\boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)}\end{bmatrix}. \tag{4.17}$$

The solution of which is

$$\begin{bmatrix}\boldsymbol{\sigma}^{(0)} \\ \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{\sigma}^{(2)}\end{bmatrix} = \left(\begin{bmatrix}\frac{1}{(n-1)vu} & 0 & 0 \\ 0 & \frac{1}{(n-1)vu(u-1)} & 0 \\ 0 & 0 & \frac{1}{(n-1)v(v-1)u^2}\end{bmatrix} \otimes \boldsymbol{I}_{\frac{m(m+1)}{2}}\right)\begin{bmatrix}\boldsymbol{r}^{(0)} \\ \boldsymbol{r}^{(1)} \\ \boldsymbol{r}^{(2)}\end{bmatrix}.$$

Now, the right hand side of the Equation (4.17) can be expressed in terms of $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ as defined in (3.5), (3.6) and (3.7) respectively, and then we have

$$\begin{bmatrix}(n-1)vu & 0 & 0 \\ 0 & (n-1)vu(u-1) & 0 \\ 0 & 0 & (n-1)v(v-1)u^2\end{bmatrix}\begin{bmatrix}\boldsymbol{\Gamma}_0 \\ \boldsymbol{\Gamma}_1 \\ \boldsymbol{\Gamma}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{C}_0 \\ \boldsymbol{C}_1 \\ \boldsymbol{C}_2\end{bmatrix}.$$

Solving this equation we get

$$\begin{aligned}\begin{bmatrix}\boldsymbol{\Gamma}_0 \\ \boldsymbol{\Gamma}_1 \\ \boldsymbol{\Gamma}_2\end{bmatrix} &= \begin{bmatrix}(n-1)vu & 0 & 0 \\ 0 & (n-1)vu(u-1) & 0 \\ 0 & 0 & (n-1)v(v-1)u^2\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{C}_0 \\ \boldsymbol{C}_1 \\ \boldsymbol{C}_2\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{(n-1)vu} & 0 & 0 \\ 0 & \frac{1}{(n-1)vu(u-1)} & 0 \\ 0 & 0 & \frac{1}{(n-1)v(v-1)u^2}\end{bmatrix}\begin{bmatrix}\boldsymbol{C}_0 \\ \boldsymbol{C}_1 \\ \boldsymbol{C}_2\end{bmatrix}.\end{aligned}$$

Therefore, estimators for $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are

$$\widetilde{\boldsymbol{\Gamma}}_0 = \frac{1}{(n-1)vu}\boldsymbol{C}_0,$$

$$\widetilde{\boldsymbol{\Gamma}}_1 = \frac{1}{(n-1)vu(u-1)}\boldsymbol{C}_1,$$

$$\widetilde{\boldsymbol{\Gamma}}_2 = \frac{1}{(n-1)v(v-1)u^2}\boldsymbol{C}_2.$$

Now using Result 2 we have the following theorem.

Theorem 2. *Assume that* $\boldsymbol{y}_{nvum\times 1} \sim N\big((\mathbf{1}_n \otimes \boldsymbol{I}_{vum})\boldsymbol{\mu}, \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}\big)$ *with DE covariance structure on* $\boldsymbol{\Gamma}$*, i.e.,*

$$\boldsymbol{\Gamma} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) + \boldsymbol{I}_v \otimes \boldsymbol{J}_u \otimes (\boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_2) + \boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_2,$$

where $\boldsymbol{\Gamma}_0$ *is* $m \times m$ *unknown positive definite and symmetric,* $\boldsymbol{\Gamma}_1$ *and* $\boldsymbol{\Gamma}_2$ *are* $m \times m$ *unknown symmetric matrices such that* $\boldsymbol{\Gamma}$ *is positive definite. Then*

$$\widetilde{\boldsymbol{\mu}} = (\widetilde{\boldsymbol{\mu}}'_{11}, \ldots, \widetilde{\boldsymbol{\mu}}'_{vu})' = \frac{1}{n}\sum_{r=1}^{n} \boldsymbol{y}_r, \tag{4.18}$$

where $\underset{nvum\times 1}{\boldsymbol{y}} = (\boldsymbol{y}'_1, \boldsymbol{y}'_2, \ldots, \boldsymbol{y}'_n)'$ *with* $\boldsymbol{y}_r = (\boldsymbol{y}'_{r,11}, \ldots, \boldsymbol{y}'_{r,vu})'$ *and*

$$\widetilde{\boldsymbol{\Gamma}} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes (\widetilde{\boldsymbol{\Gamma}}_0 - \widetilde{\boldsymbol{\Gamma}}_1) + \boldsymbol{I}_v \otimes \boldsymbol{J}_u \otimes (\widetilde{\boldsymbol{\Gamma}}_1 - \widetilde{\boldsymbol{\Gamma}}_2) + \boldsymbol{J}_v \otimes \boldsymbol{J}_u \otimes \widetilde{\boldsymbol{\Gamma}}_2, \tag{4.19}$$

where $\widetilde{\boldsymbol{\Gamma}}_0 = \frac{1}{(n-1)vu}\boldsymbol{C}_0$, $\widetilde{\boldsymbol{\Gamma}}_1 = \frac{1}{(n-1)vu(u-1)}\boldsymbol{C}_1$ *and* $\widetilde{\boldsymbol{\Gamma}}_2 = \frac{1}{(n-1)v(v-1)u^2}\boldsymbol{C}_2$ *are the best unbiased estimators (BUE) for* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Gamma}$ *respectively. Here* $\boldsymbol{C}_0$, $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *are defined in (3.5), (3.6) and (3.7) respectively.*

Proof. Our estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Gamma}$ are BLUE and BQUE, respectively. Now, because they are function of complete statistics from Result 2 it follows that they are BUE. □

The following theorem states that the estimators presented in Theorem 2 are consistent and obviously the family of distribution of the above estimators is complete.

Theorem 3. *Estimators given in (4.18) and (4.19) are consistent. Moreover, the family of distributions of these estimators is complete.*

Proof. Note that the variance of the quadratic forms $\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$, where $\boldsymbol{y} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$, is given by the following formula

$$\mathrm{var}(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) = 2\mathrm{tr}\big\{(\boldsymbol{A}\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}) + (\boldsymbol{A}\boldsymbol{V}\boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{V}\boldsymbol{A})\big\}. \tag{4.20}$$

In a special case, if $\boldsymbol{A} = \boldsymbol{M}\boldsymbol{A}\boldsymbol{M}$, and if $\boldsymbol{M}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{M}$ then $\boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}' = \mathbf{0}$, and (4.20) reduces to the following form

$$\mathrm{var}(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) = 2\mathrm{tr}(\boldsymbol{M}\boldsymbol{A}\boldsymbol{V}\boldsymbol{A}\boldsymbol{V}). \tag{4.21}$$

Now making an use of (4.21) to the DE covariance structure of the covariance matrix of $\boldsymbol{y}$ and from (4.13) it follows that for any fixed $\boldsymbol{\Gamma}$

$$
\begin{aligned}
\mathrm{var}(\widetilde{\sigma}_{ij}^{(0)}) &= 2\mathrm{tr}\Big\{\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\otimes \boldsymbol{I}_{vum}\Big]\Big[\frac{1}{(n-1)vu}(\boldsymbol{I}_{nvu}\otimes \boldsymbol{A}_{ij})(\boldsymbol{I}_n\otimes \boldsymbol{\Gamma})\frac{1}{(n-1)vu}(\boldsymbol{I}_{nvu}\otimes \boldsymbol{A}_{ij})(\boldsymbol{I}_n\otimes \boldsymbol{\Gamma})\Big]\Big\} \\
&= \frac{2}{[(n-1)vu]^2}\mathrm{tr}\Big\{\Big[\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\otimes \boldsymbol{I}_{vum}\Big]\Big[\boldsymbol{I}_n\otimes\Big((\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}(\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\Big)\Big]\Big\} \\
&= \frac{2}{[(n-1)vu]^2}\mathrm{tr}\Big\{\Big(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{J}_n\Big)\otimes\Big((\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}(\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\Big)\Big\} \\
&= \frac{2}{(n-1)v^2u^2}\mathrm{tr}\Big\{(\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}(\boldsymbol{I}_{vu}\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\Big\}.
\end{aligned}
$$

Thus, if $n\to\infty$ then

$$\mathrm{var}(\widetilde{\sigma}_{ij}^{(0)}) \to 0.$$

Similarly, from (4.14) it follows that for any fixed $\boldsymbol{\Gamma}$ if $n\to\infty$ then

$$\mathrm{var}(\widetilde{\sigma}_{ij}^{(1)}) = \frac{2}{(n-1)v^2u^2(u-1)^2}\mathrm{tr}\Big\{(\boldsymbol{I}_v\otimes(\boldsymbol{J}_u-\boldsymbol{I}_u)\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}(\boldsymbol{I}_v\otimes(\boldsymbol{J}_u-\boldsymbol{I}_u)\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\Big\}\to 0$$

and from (4.14) it follows that for any fixed $\boldsymbol{\Gamma}$ if $n\to\infty$ then

$$\mathrm{var}(\widetilde{\sigma}_{ij}^{(2)}) = \frac{2}{(n-1)v^2(v-1)^2u^4}\mathrm{tr}\Big\{((\boldsymbol{J}_v-\boldsymbol{I}_v)\otimes \boldsymbol{J}_u\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}((\boldsymbol{J}_v-\boldsymbol{I}_v)\otimes \boldsymbol{J}_u\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\Big\}\to 0.$$

To finish the proof, note that the estimators for $\boldsymbol{\mu}$ and the estimators for elements of covariance matrix are one-to-one functions of minimal sufficient statistic given by (4.14) and (4.15). □

Remark 1. *As mentioned in the Introduction for $v=1$, the data become doubly multivariate with BCS covariance structure. Now, for $v=1$, the above equations reduces to*

$$
\begin{aligned}
var(\widetilde{\sigma}_{ij}^{(0)}) &= \frac{2}{(n-1)u^2}\mathrm{tr}\big\{(\boldsymbol{I}_u\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}(\boldsymbol{I}_u\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\big\}, \\
var(\widetilde{\sigma}_{ij}^{(1)}) &= \frac{2}{(n-1)u^2(u-1)^2}\mathrm{tr}\big\{((\boldsymbol{J}_u-\boldsymbol{I}_u)\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}((\boldsymbol{J}_u-\boldsymbol{I}_u)\otimes \boldsymbol{A}_{ij})\boldsymbol{\Gamma}\big\},
\end{aligned}
$$

which are exactly same as obtained in Roy et al. (2016) for BCS covariance structure.

We conduct a study to check the behavior of $\mathrm{var}(\widetilde{\sigma}_{ij}^{(0)})$, $\mathrm{var}(\widetilde{\sigma}_{ij}^{(1)})$ and $\mathrm{var}(\widetilde{\sigma}_{ij}^{(2)})$. We consider the situation where $\boldsymbol{\Gamma}_0=\boldsymbol{I}$, an identity matrix, and $\boldsymbol{\Gamma}_1=\boldsymbol{0}$, the matrix of zeros. If $\boldsymbol{\Gamma}_0=\boldsymbol{I}$ and $\boldsymbol{\Gamma}_1=\boldsymbol{\Gamma}_2=\boldsymbol{0}$, then $\boldsymbol{\Gamma}$ is the identity matrix $\boldsymbol{\Gamma}=\boldsymbol{I}$. For the identity matrix $\boldsymbol{\Gamma}$ the formulas of variances of estimators for $\sigma^{(0)},\sigma^{(1)}$ and $\sigma^{(2)}$ are:

$$
\begin{aligned}
\mathrm{var}(\widetilde{\sigma}_{ij}^{(0)}) &= \frac{2}{(n-1)vu}, \\
\mathrm{var}(\widetilde{\sigma}_{ij}^{(1)}) &= \frac{2}{(n-1)vu(u-1)}, \quad \text{and} \\
\mathrm{var}(\widetilde{\sigma}_{ij}^{(2)}) &= \frac{2}{(n-1)v(v-1)u^2}.
\end{aligned}
$$

For graphical presentation we choose maximum value of $n = 25$, maximum value of $u = 10$ and the maximum value of $v = 10$, which are typical in most real data sets. For each figure, values for n are chosen from 3 to 25 and for u and v from 2 to 10. For the plot of n and u, v is treated as constant and $v = 2$. Similarly, For the plot of n and v, u is treated as constant and $u = 2$. The Figures 1, 2 and 3 reveal the fact that if $n \to \infty$ then variances of the estimators $\text{var}(\widetilde{\sigma}_{ij}^{(0)}) \to 0$, $\text{var}(\widetilde{\sigma}_{ij}^{(1)}) \to 0$ and $\text{var}(\widetilde{\sigma}_{ij}^{(2)}) \to 0$. In other words it means that our estimators for $\sigma^{(0)}, \sigma^{(1)}$ and $\sigma^{(2)}$ are consistent.

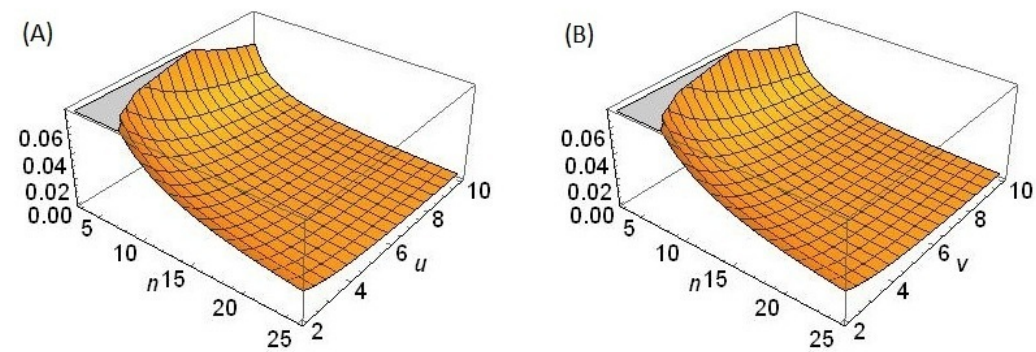

Figure 1: (A) Variance $\text{var}(\widetilde{\sigma}_{ij}^{(0)})$ for n and u, and (B) Variance $\text{var}(\widetilde{\sigma}_{ij}^{(0)})$ for n and v.

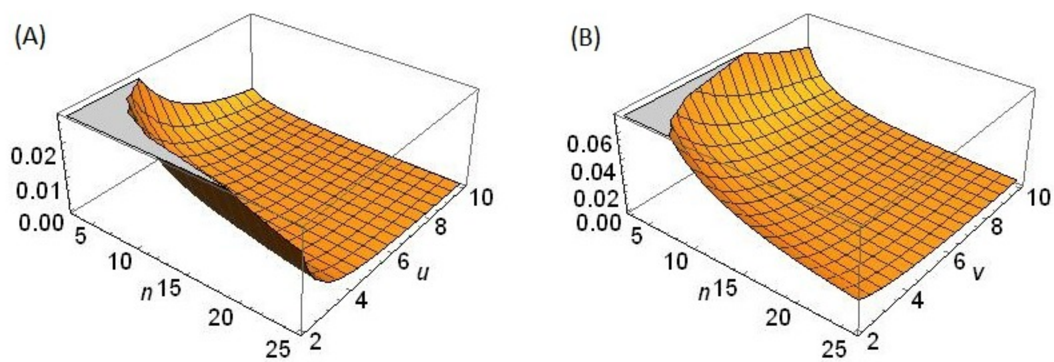

Figure 2: (A) Variance $\text{var}(\widetilde{\sigma}_{ij}^{(1)})$ for n and u, and (B) Variance $\text{var}(\widetilde{\sigma}_{ij}^{(1)})$ for n and v.

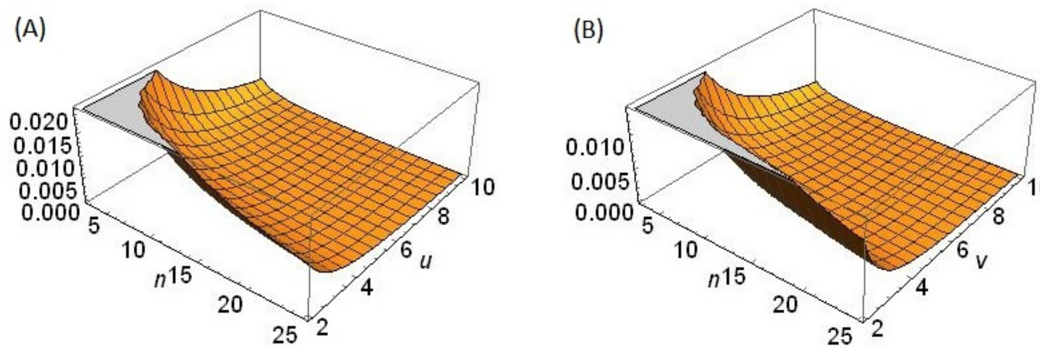

Figure 3: (A) Variance $\mathrm{var}(\widetilde{\sigma}_{ij}^{(2)})$ for n and u, and (B) Variance $\mathrm{var}(\widetilde{\sigma}_{ij}^{(2)})$ for n and v.

The following section demonstrates our methods with a real data.

5 A real data Example

Results of Section 4 are applied to the Glaucoma data that is described in the Introduction. For this data set $m = 2$, $u = 2$ and $v = 3$. Using the formula (3.4) presented in Theorem 1, the (2×1) dimensional partitioned mean vector for different $s = 1, 2$, and for different $t = 1, 2, 3$ are presented in Table 1.

Table 1 The (2×1) dimensional partitioned mean vector

t	s	$\widetilde{\boldsymbol{\mu}}_{ts}$
1	1	$(24.333, 527.367)'$
1	2	$(23.567, 534.633)'$
2	1	$(20.233, 525.333)'$
2	2	$(19.567, 532.500)'$
3	1	$(19.233, 527.133)'$
3	2	$(18.933, 534.867)'$

Using Theorem 2 we say that the above estimate $\widetilde{\boldsymbol{\mu}}$ is BLUE for $\boldsymbol{\mu}$. Additionally, using the formulas (3.8), (3.9) and (3.10) presented in Section 3 the unbiased estimates $\widetilde{\boldsymbol{\Gamma}}_0$, $\widetilde{\boldsymbol{\Gamma}}_1$ and $\widetilde{\boldsymbol{\Gamma}}_2$ are

$$\widetilde{\boldsymbol{\Gamma}}_0 = \begin{bmatrix} 12.230 & 12.061 \\ 12.061 & 426.155 \end{bmatrix}, \widetilde{\boldsymbol{\Gamma}}_1 = \begin{bmatrix} 5.826 & 6.939 \\ 6.939 & 164.156 \end{bmatrix}, \text{ and } \widetilde{\boldsymbol{\Gamma}}_2 = \begin{bmatrix} 3.528 & 9.268 \\ 9.268 & 288.684 \end{bmatrix},$$

respectively. Using the above estimates the unbiased estimate of $\mathbf{\Gamma}$ is

$$\widetilde{\mathbf{\Gamma}} = \boldsymbol{I}_v \otimes \boldsymbol{I}_u \otimes \widetilde{\mathbf{\Gamma}}_0 + \boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \widetilde{\mathbf{\Gamma}}_1 + (\boldsymbol{J}_v - \boldsymbol{I}_v) \otimes \boldsymbol{J}_u \otimes \widetilde{\mathbf{\Gamma}}_2 =$$

$$\left[\begin{array}{cccc:cccc:cccc}
12.230 & 12.061 & 5.826 & 6.939 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 \\
12.061 & 426.155 & 6.939 & 164.156 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 \\
5.826 & 6.939 & 12.230 & 12.061 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 \\
6.939 & 164.156 & 12.061 & 426.155 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 \\
\hdashline
3.528 & 9.268 & 3.528 & 9.268 & 12.230 & 12.061 & 5.826 & 6.939 & 3.528 & 9.268 & 3.528 & 9.268 \\
9.268 & 288.684 & 9.268 & 288.684 & 12.061 & 426.155 & 6.939 & 164.156 & 9.268 & 288.684 & 9.268 & 288.684 \\
3.528 & 9.268 & 3.528 & 9.268 & 5.826 & 6.939 & 12.230 & 12.061 & 3.528 & 9.268 & 3.528 & 9.268 \\
9.268 & 288.684 & 9.268 & 288.684 & 6.939 & 164.156 & 12.061 & 426.155 & 9.268 & 288.684 & 9.268 & 288.684 \\
\hdashline
3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 12.230 & 12.061 & 5.826 & 6.939 \\
9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 12.061 & 426.155 & 6.939 & 164.156 \\
3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 3.528 & 9.268 & 5.826 & 6.939 & 12.230 & 12.061 \\
9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 9.268 & 288.684 & 6.939 & 164.156 & 12.061 & 426.155
\end{array}\right].$$

Using Theorems 2 and 3 we say that the above estimate $\widetilde{\mathbf{\Gamma}}$ is the best unbiased, consistent and complete estimate of the DE covariance structure $\mathbf{\Gamma}$.

6 Conclusions

The obtained results in this paper demonstrate the optimality of estimates for both fixed effects and DE covariance matrices using the coordinate free approach theory for a model with DE covariance structure. It thus provides a valuable alternative to maximum likelihood estimation, taking as a base for estimation the algebraic structure of the model. Another significative property is the unbiasedness of the proposed estimates, a property that maximum likelihood does not, in general, guarantees or aims at.

Acknowledgements

The authors thank the Mathematical Research and Conference Center of the Polish Academy of Sciences Bedlewo, Poland, for giving the opportunity and support for this paper in 2015. The fifth author acknowledges that his work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

References

[1] C. A. Coelho and A. Roy, *Testing the hypothesis of a doubly exchangeable covariance matrix for elliptically contoured distributions*, Working Paper No. 0002MSS-253-2014, College of Business, University of Texas at San Antonio, 2014.

[2] M. Fonseca, J.T. Mexia and R. Zmyślony, *Least squares and generalized least squares in models with orthogonal block structure*. Journal of Statistical Planning and Inference, 140(2010), 1346-1352.

[3] H. Drygas, *The Coordinate-Free Approach to Gauss-Markov Estimation*, Springer, Berlin, Heidelberg, 1970.

[4] S. Gnot, W. Klonecki and R. Zmyślony, *Uniformly minimum variance unbiased estimation in Euclidean vector spaces*, Bull. de l. Academie Polon. des Sciences XXIV 4(1976), pp. 281-286.

[5] S. Gnot, W. Klonecki and R. Zmyślony, *Uniformly minimum variance unbiased estimation in various classes of estimators, Math. Operationsforsch.* Statist.,Ser. Statistics 8(1977), pp. 199-210.

[6] S. Gnot, W. Klonecki and R. Zmyślony, *Best unbiased estimation: a coordinate free-approach.* Probability and Statistics, 1(1980), pp. 1-13.

[7] P. Jordan, von, J. Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism.* The Annals of Mathematics, 35(1934), pp. 29-64.

[8] A. Kozioł, A. Roy, R. Zmyślony, R. Leiva and M. Fonseca, *Best Unbiased estimators for doubly multivariate data*. Submitted, 2015.

[9] W. Kruskal, *When are Gauss-Markov and Least Squares Estimators Identical? A Coordinate-Free Approach. The Annals of Mathematical Statistics*, 39(1968), pp.70-75.

[10] R. Leiva and A. Roy, *Classification rules for triply multivariate data with an ar(1) correlation structure on the repeated measures over time*, Journal of Statistical Planning and Inference, 139(2009), pp. 2598-2613

[11] R. Leiva and A. Roy, *Linear discrimination for multi-level multivariate data with separable means and jointly equicorrelated covariance structure*. Journal of Statistical Planning and Inference. 141(2011), pp. 1910-1924.

[12] R. Leiva and A. Roy, *Linear discrimination for three-level multivariate data with separable additive mean vector and doubly exchangeable covariance structure*, Computational Statistics and Data Analysis, 56(2012), pp. 1644-1661.

[13] A. Roy, *A two-stage principal component analysis of symbolic data using equicorrelated and jointly equicorrelated covariance structures*, Working Paper SERIES: WP # 0006MSS-253-2014, College of Business, The university of Texas at San Antonio, 2014.

[14] A. Roy and R. Leiva, *discrimination with jointly equicorrelated multi-level multivariate data*, Advances in Data Analysis and Classification, 1(2007), pp. 175-199.

[15] A. Roy and R. Leiva, *Estimating and testing a structured covariance matrix for three-level multivariate data*, Communications in Statistics - Theory and Methods, 40(2011), pp. 1945-1963.

[16] A. Roy and M. Fonseca, *Linear models with doubly exchangeable distributed errors*, Comm. Statist. Theory Methods, 41(2012), pp. 2545–2569.

[17] A. Roy, R. Zmyślony, M. Fonseca and R. Leiva, *Optimal estimation for doubly multivariate data in blocked compound symmetric covariance structure*, Journal of Multivariate Analysis, 144(2016), pp. 81-90.

[18] J.F. Seely, *Quadratic subspaces and completeness*. The Annals of Mathematical Statistics, 42(1971), pp. 710-721.

[19] J.F. Seely, *Minimal sufficient statistics and completeness for multivariate normal families*. Sankhya (Statistics). The Indian Journal of Statistics. Series A, 39(1977), pp. 170-185.

[20] R. Zmyślony, *On estimation of parameters in linear models*, Applicationes Mathematicae XV 3(1976), pp. 271-276.

[21] R. Zmyślony, *A characterization of best linear unbiased estimators in the general linear model*, Lecture Notes in Statistics, 2(1978), pp. 365-373.

[22] R. Zmyślony, *Completeness for a family of normal distributions*, Mathematical Statistics, Banach Center Publications 6(1980), pp. 355-357.