THE UNIVERSITY OF TEXAS AT SAN ANTONIO, COLLEGE OF BUSINESS

Working Paper SERIES

Date January 23, 2014

WP # 0002MSS-253-2014

Testing the hypothesis of a doubly exchangeable covariance matrix for elliptically contoured distributions

Carlos A. Coelho
Departamento de Matemática
Faculdade de Ciencias e Tecnologia
Universidade Nova de Lisboa

Anuradha Roy
Department of Management Science and Statistics
The University of Texas at San Antonio

ONE UTSA CIRCLE
SAN ANTONIO, TEXAS 78249-0631
210 458-4317 | BUSINESS.UTSA.EDU

Testing the hypothesis of a doubly exchangeable covariance matrix for elliptically contoured distributions

Carlos A. Coelho[a,b], Anuradha Roy[c,*]

[a]*Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa*
[b]*CMA-FCT/UNL - Centro de Matemática e Aplicações, Universidade Nova de Lisboa, Portugal*
[c]*Department of Management Science and Statistics, The University of Texas at San Antonio, San Antonio, Texas, U.S.A.*

Abstract

In this paper the authors study the problem of testing the hypothesis of a doubly exchangeable covariance matrix for three-level multivariate observations, taken on m variables over u sites and over v time/spatial points. Through the decomposition of the main hypothesis into a set of three sub-hypotheses, the likelihood ratio test statistic is defined, its exact moments determined, and its exact distribution studied. Because this distribution is very much intricate, a very precise near-exact distribution is developed. Numerical studies conducted to evaluate the closeness between this near-exact distribution and the exact distribution show the very good performance of this approximation even for very small sample sizes. A simulation study is also conducted and a real data example is presented.

Keywords: characteristic function, composition of hypotheses, distribution of likelihood ratio statistics, product of independent Beta random variables, sum of independent Gamma random variables.

JEL Classification: C12

1. Introduction

Advances in computing power in the past few decades greatly encouraged the collection of multi-level multivariate data in all fields of science: biomedical, medical, social science, engineering and business. And, with these data sets complex multivariate testing problems occur frequently. It is common in clinical trial studies to collect measurements on more than one response variable at several locations taken repeatedly over time on one experimental unit to test the effectiveness of some medication, diet or treatment. These are called three-level multivariate data. For example consider an example from a clinical trial of osteoporosis. Osteoporosis or porous bone is an age-related disorder involving in a progressive decrease in bone mass due to the loss of minerals — mainly calcium. As a result, bones become weakened and more susceptible to fractures. Currently it is estimated that one of every four post-menopausal women has osteoporosis. Although it is more common in white or Asian women older than 50 years, osteoporosis can occur in almost any person at any age: osteoporosis is not just an 'old womans disease'. In fact, more than 2 million American men have osteoporosis. The estimated national cost for osteoporosis and related injuries

*Corresponding author: Department of Management Science and Statistics, The University of Texas at San Antonio, San Antonio, Texas, U.S.A.; phone: +1 210 458 6343, Fax: +1 210 458 6350.
Email addresses: cmac@fct.unl.pt (Carlos A. Coelho), Anuradha.Roy@utsa.edu (Anuradha Roy)

January 15, 2014

is \$14 billion each year in the United States. Johnson and Wichern (2007) report data on a study where an investigator measures the mineral content of three bones, radius, humerus and ulna by photon absorptiometry to examine whether a particular dietary supplement increase bone mineral content and mass in older women. All three measurements are also recorded on the dominant and non-dominant sides for each woman. These doubly multivariate measurements are taken on 25 women. The bone mineral contents for the first 24 women are measured after one year of their participation in the experimental program. Thus, for our analysis we take only the first 24 women in the first data set, and combine these two data sets side by side into a new one. Thus, this new data set has a three-level multivariate structure, with $m = 3$ variables, for $u = 2$ locations, over $v = 2$ time points. We want to test whether this new three-level multivariate data set has doubly exchangeable covariance structure to fit a linear model (Roy and Fonseca, 2012) on this new data set. This fact motivated us to develop a new method of testing for a doubly exchangeable covariance matrix for three-level multivariate data. Hypothesis testing on three-level multivariate data was first studied by Roy and Leiva (2008). These two authors introduced parametrically parsimonious models for hypotheses testing problems by using a "Blocked compound symmetric" covariance structure on the measurement vector over sites in addition to either an autoregressive of order one (AR(1)) or a compound symmetry (CS) correlation structure on the spatial repeated measurements. To the best of the authors' knowledge, tests of hypotheses for doubly exchangeable covariance matrix for three-level multivariate data have not yet been studied.

Let $\boldsymbol{y}$ be the muv-variate partitioned real-valued random vector of all measurements. We partition this vector $\boldsymbol{y}$ as follows:

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_v \end{pmatrix}, \qquad \text{where} \quad \boldsymbol{y}_t = \begin{pmatrix} \boldsymbol{y}_{t1} \\ \vdots \\ \boldsymbol{y}_{tu} \end{pmatrix}, \qquad \text{with} \quad \boldsymbol{y}_{ts} = \begin{pmatrix} \boldsymbol{y}_{ts1} \\ \vdots \\ \boldsymbol{y}_{tsm} \end{pmatrix},$$

for $s = 1, \ldots, u$, $t = 1, \ldots, v$. The m-dimensional vector of measurements $\boldsymbol{y}_{ts}$ represents the replicate on the sth location and at the tth time point.

Let $\boldsymbol{\Theta} = \text{Cov}\,[\boldsymbol{y}]$ be the $(muv \times muv)-$dimensional partitioned covariance matrix with $\text{Cov}\,[\boldsymbol{y}] = \left(\boldsymbol{\Theta}_{\boldsymbol{y}_t, \boldsymbol{y}_{t^*}}\right) = \left(\boldsymbol{\Theta}_{tt^*}\right)$, and $\boldsymbol{\Theta}_{tt^*} = \left(\boldsymbol{\Theta}_{\boldsymbol{y}_{ts}, \boldsymbol{y}_{t^*s^*}}\right) = \left(\boldsymbol{\Theta}_{ts,t^*s^*}\right)$, where $\boldsymbol{\Theta}_{tt^*} = \text{Cov}\,[\boldsymbol{y}_t, \boldsymbol{y}_{t^*}]$ and $\boldsymbol{\Omega}_{ts,t^*s^*} = \text{Cov}[\boldsymbol{y}_{ts}, \boldsymbol{y}_{t^*s^*}]$, for $t, t^* = 1, \ldots, v$ and $s, s^* = 1, \ldots, u$.

We say that a covariance matrix has a Doubly exchangeable covariance structure (Roy and Leiva, 2007) if it can be written as

$$\boldsymbol{\Theta} = \left[\begin{array}{cccc|cccc|c|cccc}
\boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0 & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\hline
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0 & \cdots & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\
\hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\hline
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_1 \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_1 & \boldsymbol{U}_0 & \cdots & \boldsymbol{U}_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{W} & \boldsymbol{W} & \cdots & \boldsymbol{W} & \cdots & \boldsymbol{U}_1 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_0
\end{array}\right],$$

$$\begin{aligned}
&= \boldsymbol{I}_{uv} \otimes \boldsymbol{U}_0 + [\boldsymbol{I}_v \otimes (\boldsymbol{J}_u - \boldsymbol{I}_u))] \otimes \boldsymbol{U}_1 + [\boldsymbol{J}_{uv} - (\boldsymbol{I}_v \otimes \boldsymbol{J}_u)] \otimes \boldsymbol{W} \\
&= \boldsymbol{I}_{uv} \otimes (\boldsymbol{U}_0 - \boldsymbol{U}_1) + \boldsymbol{I}_v \otimes \mathbf{J}_u \otimes (\boldsymbol{U}_1 - \boldsymbol{W}) + \boldsymbol{J}_{uv} \otimes \boldsymbol{W}, \qquad (1.1)
\end{aligned}$$

where $\boldsymbol{U}_0$ is a positive definite symmetric $m \times m$ matrix, and $\boldsymbol{U}_1$ and $\boldsymbol{W}$ are symmetric $m \times m$ matrices. The matrices $\boldsymbol{U}_0, \boldsymbol{U}_1$ and $\boldsymbol{W}$ are all unstructured.

Thus, the vectors $\boldsymbol{y}_{11}, \ldots, \boldsymbol{y}_{1u}, \ldots, \boldsymbol{y}_{v1}, \ldots, \boldsymbol{y}_{vu}$ are doubly exchangeable if

$$\mathrm{Cov}\left[\boldsymbol{y}_{ts}; \boldsymbol{y}_{t^*s^*}\right] = \begin{cases} \boldsymbol{U}_0 & \text{if} \quad t = t^* \quad \text{and} \quad s = s^*, \\ \boldsymbol{U}_1 & \text{if} \quad t = t^* \quad \text{and} \quad s \neq s^*, \\ \boldsymbol{W} & \text{if} \quad t \neq t^*. \end{cases}$$

The $m \times m$ diagonal blocks $\boldsymbol{U}_0$ in (1.1) represent the variance-covariance matrix of the m response variables at any given location and at any given time point, whereas the $m \times m$ off-diagonal blocks $\boldsymbol{U}_1$ in (1.1) represent the covariance matrix of the m response variables between any two locations and at any given time point. We assume $\boldsymbol{U}_0$ is constant for all locations and time points, and $\boldsymbol{U}_1$ is same for all location pairs and for all time points. The $m \times m$ off-diagonal blocks $\boldsymbol{W}$ represent the covariance matrix of the m response variables between any two time points. It is assumed to be the same for any pair of time points, irrespective of location or between any two locations.

2. Formulation of the hypothesis and the likelihood ratio test

Let $\boldsymbol{y} \sim N_{muv}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We are interested in testing the hypothesis

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Theta}\,, \qquad (2.1)$$

where $\boldsymbol{\Theta}$ is defined in (1.1).

In Lemma 3.1 in Roy and Fonseca (2012), it is shown that we may write

$$\boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}^{*}\boldsymbol{\Theta}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Gamma}^{\bullet\prime} = \begin{bmatrix} \boldsymbol{\Delta}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Delta}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I}_{u-1}\otimes\boldsymbol{\Delta}_1 \end{bmatrix},$$

where $\boldsymbol{\Gamma}^{*} = \underset{v\times v}{\boldsymbol{C}'} \otimes \boldsymbol{I}_{mu}$ and $\boldsymbol{\Gamma}^{\bullet} = \boldsymbol{I}_v \otimes (\underset{u\times u}{\boldsymbol{C}^{*\prime}} \otimes \boldsymbol{I}_m)$, where $\boldsymbol{C}$ and $\boldsymbol{C}^{*}$ are orthogonal matrices whose first columns are proportional to $\mathbf{1}$'s, so that $\boldsymbol{\Gamma}^{\bullet}$ and $\boldsymbol{\Gamma}^{*}$ are not function of either $\boldsymbol{U}_0$, or $\boldsymbol{U}_1$ or $\boldsymbol{W}$, and

$$\begin{aligned} \boldsymbol{\Delta}_1 &= \boldsymbol{U}_0 - \boldsymbol{U}_1, \\ \boldsymbol{\Delta}_2 &= \boldsymbol{U}_0 + (u-1)\,\boldsymbol{U}_1 - u\,\boldsymbol{W} = (\boldsymbol{U}_0 - \boldsymbol{U}_1) + u\,(\boldsymbol{U}_1 - \boldsymbol{W})\,, \\ \text{and} \quad \boldsymbol{\Delta}_3 &= \boldsymbol{U}_0 + (u-1)\,\boldsymbol{U}_1 + u\,(v-1)\,\boldsymbol{W} = (\boldsymbol{U}_0 - \boldsymbol{U}_1) + u\,(\boldsymbol{U}_1 - \boldsymbol{W}) + uv\boldsymbol{W}. \end{aligned}$$

Thus, to test H_0 in (2.1) is equivalent to test

$$H_0 : \boldsymbol{\Sigma}^{*} = \boldsymbol{\Omega} \tag{2.2}$$

where

$$\boldsymbol{\Sigma}^{*} = \boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}^{*}\boldsymbol{\Sigma}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Gamma}^{\bullet\prime} \qquad \text{and} \qquad \boldsymbol{\Omega} = \boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}^{*}\boldsymbol{\Theta}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Gamma}^{\bullet\prime}.$$

We may split the null hypothesis in (2.2) as

$$H_0 \equiv \left(H_{0c|a} \,||\, H_{0b|a}\right) \circ H_{0a}\,, \tag{2.3}$$

where '$\circ$' means 'after' and '$||$' means 'parallel', meaning 'either after or before'.

In (2.3),

$$H_{0a} : \boldsymbol{\Sigma}^{*} = \text{block-diag}(\boldsymbol{\Sigma}_i^{*},\, i = 1, \ldots, uv)\,, \tag{2.4}$$

is the hypothesis of independence of the uv diagonal blocks $\boldsymbol{\Sigma}_i^{*}$ $(i = 1, \ldots, uv)$ of size $m\times m$ of $\boldsymbol{\Sigma}^{*}$;

$$H_{0b|a}: \quad \underbrace{\boldsymbol{\Sigma}_2^{*} = \cdots = \boldsymbol{\Sigma}_u^{*}}_{u-1} = \underbrace{\boldsymbol{\Sigma}_{u+2}^{*} = \cdots = \boldsymbol{\Sigma}_{2u}^{*}}_{u-1} = \cdots = \underbrace{\boldsymbol{\Sigma}_{(v-1)u+2}^{*} = \cdots = \boldsymbol{\Sigma}_{vu}^{*}}_{u-1}, \quad \text{assuming } H_{0a}, \tag{2.5}$$

is the hypothesis of equality of $v(u-1)$ covariance matrices of dimension $m\times m$, assuming H_{0a}, and

$$H_{0c|a}: \quad \boldsymbol{\Sigma}_{u+1}^{*} = \boldsymbol{\Sigma}_{2u+1}^{*} = \cdots = \boldsymbol{\Sigma}_{(v-1)u+1}^{*} \quad \text{assuming } H_{0a}, \tag{2.6}$$

is the hypothesis of equality of the covariance matrices $\boldsymbol{\Sigma}_{u+1}^{*}, \boldsymbol{\Sigma}_{2u+1}^{*}, \ldots, \boldsymbol{\Sigma}_{(v-1)u+1}^{*}$, assuming H_{0a}.

The likelihood ratio test (l.r.t.) statistic to test H_{0a} in (2.4) is (Anderson, 2003, Sec. 9.2)

$$\Lambda_a = \left(\frac{|\boldsymbol{A}|}{\prod_{j=1}^{uv} |\boldsymbol{A}_j|}\right)^{n/2}$$

where $\boldsymbol{A} = \boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}^{*}\boldsymbol{A}^{+}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Gamma}^{\bullet\prime}$ is the is the maximum likelihood estimator (m.l.e.) of $\boldsymbol{\Sigma}^*$, and $\boldsymbol{A}_j$ its j-th diagonal $m{\times}m$ block, being $\boldsymbol{A}^+$ the m.l.e. of $\boldsymbol{\Sigma}$.

The l.r.t. statistic to test $H_{0b|a}$ in (2.5) is (Anderson, 2003, Sec. 10.2)

$$\Lambda_b = \left((v(u-1))^{mv(u-1)} \frac{\prod_{\ell=1}^{v}\prod_{k=1}^{u-1} \left|\boldsymbol{A}_{(\ell-1)u+1+k}\right|}{|\boldsymbol{A}^*|^{v(u-1)}} \right)^{n/2}, \tag{2.7}$$

where

$$\boldsymbol{A}^* = \sum_{\ell=1}^{v}\sum_{k=1}^{u-1} \boldsymbol{A}_{(\ell-1)u+1+k}\,.$$

The l.r.t. statistic to test $H_{0c|a}$ in (2.6) is (Anderson, 2003, Sec. 10.2)

$$\Lambda_c = \left((v-1)^{m(v-1)} \frac{\prod_{k=1}^{v-1} |\boldsymbol{A}_{ku+1}|}{|\boldsymbol{A}^{**}|^{v-1}} \right)^{n/2} \tag{2.8}$$

where

$$\boldsymbol{A}^{**} = \sum_{k=1}^{v-1} \boldsymbol{A}_{ku+1}\,.$$

Then, through an extension of Lemma 10.3.1 in (Anderson, 2003, Sec. 10.3), the l.r.t. statistic to test H_0 in (2.2) will be

$$\begin{aligned}\Lambda &= \Lambda_a\Lambda_b\Lambda_c \\ &= \left((v(u-1))^{mv(u-1)}\,(v-1)^{m(v-1)} \frac{|\boldsymbol{A}|}{|\boldsymbol{A}_1||\boldsymbol{A}^*|^{v(u-1)}\,|\boldsymbol{A}^{**}|^{v-1}} \right)^{n/2},\end{aligned} \tag{2.9}$$

with

$$E\left(\Lambda^h\right) = E\left(\Lambda_a^h\right) E\left(\Lambda_b^h\right) E\left(\Lambda_c^h\right), \tag{2.10}$$

since on one hand, under H_{0a} Λ_a is independent of $\prod_{j=1}^{uv} |\boldsymbol{A}_j|$ (Marques and Coelho, 2012; Coelho and Marques, 2012b), which makes Λ_a independent of Λ_b and Λ_c, while on the other hand, the $\boldsymbol{A}_j$ $(j = 1, \ldots, uv)$, under H_{0a}, are independent among themselves, which makes Λ_b and Λ_c independent because they are built on different $\boldsymbol{A}_j$'s.

In the following section we obtain the expressions for the moments of all three l.r.t. statistics Λ_a, Λ_b and Λ_c, as well as their distributions.

3. On the exact distribution of the l.r.t. statistic

Using the results in Coelho (2004), Coelho, Arnold and Marques (2010) and Marques, Coelho and Arnold (2011) we may write the h-th moment of Λ_a as

$$\begin{aligned} E\left(\Lambda_a^h\right) &= \prod_{k=1}^{uv-1}\prod_{j=1}^{m} \frac{\Gamma\left(\frac{n-j}{2}\right)\Gamma\left(\frac{n-(uv-k)m-j}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-(uv-k)m-j}{2}\right)\Gamma\left(\frac{n-j}{2}+\frac{n}{2}h\right)} \\ &= \underbrace{\left\{\prod_{j=3}^{muv}\left(\frac{n-j}{n}\right)^{r_j}\left(\frac{n-j}{n}+h\right)^{-r_j}\right\}}_{\Phi_{a,1}(h)} \underbrace{\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2}+\frac{n}{2}h\right)\Gamma\left(\frac{n-2}{2}\right)}\right)^{k^*}}_{\Phi_{a,2}(h)} \end{aligned} \tag{3.1}$$

where

$$k^* = \begin{cases} \left\lfloor \dfrac{uv}{2} \right\rfloor & m \text{ odd} \\ 0 & m \text{ even}, \end{cases} \tag{3.2}$$

and

$$r_j = \begin{cases} h_{j-2} + (-1)^j k^* & j = 3, 4 \\ r_{j-2} + h_{j-2} & j = 5, \dots, muv \,, \end{cases}$$

with

$$h_j = \begin{cases} uv - 1 & j = 1, \dots, m \\ -1 & j = m+1, \dots, muv - 2 \,. \end{cases}$$

Now, using the results in (Coelho and Marques, 2012a; Coelho, Arnold and Marques, 2010; Marques, Coelho and Arnold, 2011) we obtain the expression for the h-th moment of Λ_b as

$$\begin{aligned} E\left(\Lambda_b^h\right) &= \prod_{j=1}^{m} \prod_{k=1}^{v(u-1)} \frac{\Gamma\left(\frac{n-1}{2} - \frac{j-1}{2v(u-1)} + \frac{k-1}{v(u-1)}\right) \Gamma\left(\frac{n-j}{2} + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2} - \frac{j-1}{2v(u-1)} + \frac{k-1}{v(u-1)} + \frac{n}{2}h\right) \Gamma\left(\frac{n-j}{2}\right)} \\ &= \underbrace{\left\{ \prod_{j=2}^{m} \left(\frac{n-j}{n}\right)^{s_j} \left(\frac{n-j}{n} + h\right)^{-s_j} \right\}}_{\Phi_{b,1}(h)} \\ &\quad \underbrace{\times \left\{ \prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v(u-1)} \frac{\Gamma\left(n - 1 + \frac{k-2j}{v(u-1)}\right) \Gamma\left(n - 1 + \left\lfloor \frac{k-2j}{v(u-1)} \right\rfloor + nh\right)}{\Gamma\left(n - 1 + \frac{k-2j}{v(u-1)} + nh\right) \Gamma\left(n - 1 + \left\lfloor \frac{k-2j}{v(u-1)} \right\rfloor\right)} \right\} \times \left\{ \prod_{k=1}^{v(u-1)} \frac{\Gamma\left(\frac{n-m}{2} + \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)}\right) \Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} \right\rfloor + \frac{n}{2}h\right)}{\Gamma\left(\frac{n-m}{2} + \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} + \frac{n}{2}h\right) \Gamma\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} \right\rfloor\right)} \right\}^{m \perp\!\!\!\perp 2}}_{\Phi_{b,2}(h)} \end{aligned} \tag{3.3}$$

where s_j $(j = 2, \dots, m)$ are given in Appendix A.

By looking at (2.7) and (2.8) we may see that the h-th moment of Λ_c may be obtained, as follows, from the h-th moment of Λ_b by first replacing v by 1 and then replacing u by v,

$$
\begin{aligned}
E\left(\Lambda_c^h\right) &= \prod_{j=1}^{m}\prod_{k=1}^{v-1}\frac{\Gamma\left(\frac{n-1}{2}-\frac{j-1}{2(v-1)}+\frac{k-1}{v-1}\right)\Gamma\left(\frac{n-j}{2}+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-1}{2}-\frac{j-1}{2(v-1)}+\frac{k-1}{v-1}+\frac{n}{2}h\right)\Gamma\left(\frac{n-j}{2}\right)}\\
&= \underbrace{\left\{\prod_{j=2}^{m}\left(\frac{n-j}{n}\right)^{\delta_j}\left(\frac{n-j}{n}+h\right)^{-\delta_j}\right\}}_{\Phi_{c,1}(h)}\\
&\quad \underbrace{\times\left\{\prod_{j=1}^{\lfloor m/2\rfloor}\prod_{k=1}^{v-1}\frac{\Gamma\left(n-1+\frac{k-2j}{v-1}\right)\Gamma\left(n-1+\left\lfloor\frac{k-2j}{v-1}\right\rfloor+nh\right)}{\Gamma\left(n-1+\frac{k-2j}{v-1}+nh\right)\Gamma\left(n-1+\left\lfloor\frac{k-2j}{v-1}\right\rfloor\right)}\right\}
\times\left\{\prod_{k=1}^{v-1}\frac{\Gamma\left(\frac{n-m}{2}+\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}\right)\Gamma\left(\frac{n-m}{2}+\left\lfloor\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}\right\rfloor+\frac{n}{2}h\right)}{\Gamma\left(\frac{n-m}{2}+\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}+\frac{n}{2}h\right)\Gamma\left(\frac{n-m}{2}+\left\lfloor\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}\right\rfloor\right)}\right\}^{m\perp\!\!\!\perp 2}}_{\Phi_{c,2}(h)}
\end{aligned}
\tag{3.4}
$$

where the shape parameters δ_j $(j=2,\ldots,m)$ are given in Appendix A.

Since the supports of Λ_a, Λ_b and Λ_c are delimited, their distributions are defined by their moments, and as such, from the first expression in (3.1) we may write

$$
\Lambda_a \overset{st}{\sim} \prod_{j=1}^{m}\prod_{k=1}^{uv-1}\left(X_{jk}\right)^{n/2}, \qquad \text{where} \qquad X_{jk}\sim Beta\left(\frac{n-(uv-k)m-j}{2},\frac{(uv-k)m}{2}\right), \tag{3.5}
$$

where '$\overset{st}{\sim}$' means 'stochastically equivalent' and X_{jk} $(j=1,\ldots,m;k=1,\ldots,uv-1)$ are independent random variables, while from the first expression in (3.3) we may write

$$
\Lambda_b \overset{st}{\sim} \prod_{j=1}^{m}\prod_{k=1}^{v(u-1)}\left(X^*_{jk}\right)^{n/2}, \qquad \text{where} \qquad X^*_{jk}\sim Beta\left(\frac{n-j}{2},\frac{j-1}{2}+\frac{2k-j-1}{2v(u-1)}\right), \tag{3.6}
$$

where X^*_{jk} $(j=1,\ldots,m;k=1,\ldots,v(u-1))$ are independent, and from the first expression in (3.4) we may write

$$
\Lambda_c \overset{st}{\sim} \prod_{j=1}^{m}\prod_{k=1}^{v-1}\left(X^{**}_{jk}\right)^{n/2}, \qquad \text{where} \qquad X^{**}_{jk}\sim Beta\left(\frac{n-j}{2},\frac{j-1}{2}+\frac{2k-j-1}{2(v-1)}\right), \tag{3.7}
$$

where X^{**}_{jk} $(j=1,\ldots,m;k=1,\ldots,v-1)$ are independent, so that we may write for the overall l.r.t. statistic for H_0 in (2.3)

$$
\Lambda \overset{st}{\sim} \prod_{j=1}^{m}\left\{\left(\prod_{k=1}^{uv-1}X_{jk}\right)^{n/2}\times\left(\prod_{k=1}^{v(u-1)}X^*_{jk}\right)^{n/2}\times\left(\prod_{k=1}^{v-1}X^{**}_{jk}\right)^{n/2}\right\}, \tag{3.8}
$$

where all random variables are independent.

On the other hand, based on the results in Appendix B and from the second expressions in (3.1)-(3.4) we may respectively write, for Λ_a,

$$\Lambda_a \stackrel{st}{\sim} \left(\prod_{j=3}^{muv} e^{-Z_j}\right) \times \left(\prod_{j=1}^{k^*} (W_j)^{n/2}\right) \tag{3.9}$$

where

$$Z_j \sim \Gamma\left(r_j, \frac{n-j}{n}\right) \quad \text{and} \quad W_j \sim Beta\left(\frac{n-2}{2}, \frac{1}{2}\right) \tag{3.10}$$

are all independent random variables, while for Λ_b we may write

$$\Lambda_b \stackrel{st}{\sim} \left(\prod_{j=2}^{m} e^{-Z_j^*}\right) \times \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v(u-1)} (W_{1jk}^*)^n\right) \times \left(\prod_{k=1}^{v(u-1)} (W_{2k}^*)^{n/2}\right)^{m \perp\!\!\!\perp 2} \tag{3.11}$$

where

$$Z_j^* \sim \Gamma\left(s_j, \frac{n-j}{n}\right), \quad W_{1jk}^* \sim Beta\left(n-1+\left\lfloor \frac{k-2j}{v(u-1)} \right\rfloor, \frac{k-2j}{v(u-1)} - \left\lfloor \frac{k-2j}{v(u-1)} \right\rfloor\right), \tag{3.12}$$

and

$$W_{2k}^* \sim Beta\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} \right\rfloor, \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} - \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2v(u-1)} \right\rfloor\right) \tag{3.13}$$

are all independent random variables, while for Λ_c we may write

$$\Lambda_c \stackrel{st}{\sim} \left(\prod_{j=2}^{m} e^{-Z_j^{**}}\right) \times \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v-1} (W_{1jk}^{**})^n\right) \times \left(\prod_{k=1}^{v-1} (W_{2k}^{**})^{n/2}\right)^{m \perp\!\!\!\perp 2} \tag{3.14}$$

where

$$Z_j^{**} \sim \Gamma\left(\delta_j, \frac{n-j}{n}\right), \quad W_{1jk}^{**} \sim Beta\left(n-1+\left\lfloor \frac{k-2j}{v-1} \right\rfloor, \frac{k-2j}{v-1} - \left\lfloor \frac{k-2j}{v-1} \right\rfloor\right), \tag{3.15}$$

and

$$W_{2k}^{**} \sim Beta\left(\frac{n-m}{2} + \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2(v-1)} \right\rfloor, \frac{m-1}{2} + \frac{2k-m-1}{2(v-1)} - \left\lfloor \frac{m-1}{2} + \frac{2k-m-1}{2(v-1)} \right\rfloor\right) \tag{3.16}$$

are all independent random variables.

Thus, we have the following Theorem.

Theorem 1. *The exact distribution of the overall l.r.t. statistic* Λ *in* (2.9)*, to test* H_0 *in* (2.2) *or* (2.3) *is, for general* u*,* v *and* m*, the same as that of*

$$\left(\prod_{j=2}^{muv} e^{-T_j}\right) \times \left(\prod_{j=1}^{k^*} W_j\right)^{n/2} \times \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v(u-1)} W_{1jk}^*\right)^n \times \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v-1} W_{1jk}^{**}\right)^n \times \left\{\left(\prod_{k=1}^{v(u-1)} W_{2k}^*\right)^{n/2} \times \left(\prod_{k=1}^{v-1} W_{2k}^{**}\right)^{n/2}\right\}^{m \perp\!\!\!\perp 2} \tag{3.17}$$

where, for $j = 2, \ldots, muv$,

$$T_j \sim \Gamma\left(\mu_j, \frac{n-j}{n}\right),$$

with

$$\mu_j = \sum_{j=2}^{muv} \left(r_j^+ + s_j^+ + \delta_j^+\right) \tag{3.18}$$

where

$$r_j^+ = \begin{cases} 0 & j = 2 \\ r_j & j = 3, \ldots, muv \end{cases}$$

and

$$\omega_j^+ = \begin{cases} \omega_j & j = 2, \ldots, m \\ 0 & j = m+1, \ldots, muv \end{cases}$$

with $\omega \equiv s$ *or* $\omega \equiv \delta$, *and where the shape parameters* s_j *and* δ_j *are defined in Appendix A. The distributions of* W_j, W_{1jk}^*, W_{2k}^*, W_{1jk}^{**} *and* W_{2k}^{**} *are defined in* (3.10), (3.12), (3.13), (3.15) *and* (3.16).

Proof. The proof of the above theorem is rather trivial, from the previously established results. We may only remark that the random variables T_j are the sum of the random variables Z_j, Z_j^* and Z_j^{**}, which are independent Gamma distributed random variables, all with the same rate parameters $\frac{n-j}{n}$, for a given j. As such, for a given j, their sum is a Gamma distributed random variable with that same rate parameter and a shape parameter which is the sum of the original shape parameters. □

The following Corollary refers to the particular cases for $v = 1$ and $v = 2$, which are particular cases of interest.

Corollary 1. *For both* $v = 1$ *and* $v = 2$, *the hypothesis* $H_{0c|a}$ *vanishes and as such the distribution of* Λ *is in this case the same as that of*

$$\left(\prod_{j=2}^{muv} e^{-T_j}\right) \times \left(\prod_{j=1}^{k^*} W_j\right)^{n/2} \times \left(\prod_{j=1}^{\lfloor m/2 \rfloor} \prod_{k=1}^{v(u-1)} W_{1jk}^*\right)^n \times \left\{\left(\prod_{k=1}^{v(u-1)} W_{2k}^*\right)^{n/2}\right\}^{m \perp\!\!\!\perp 2} \tag{3.19}$$

where, for $j = 2, \ldots, muv$,

$$T_j \sim \Gamma\left(\mu_j^*, \frac{n-j}{n}\right),$$

with

$$\mu_j^* = \sum_{j=2}^{muv} \left(r_j^+ + s_j^+\right) \tag{3.20}$$

where

$$r_j^+ = \begin{cases} 0 & j = 2 \\ r_j & j = 3, \ldots, muv \end{cases} \quad \text{and} \quad s_j^+ = \begin{cases} s_j & j = 2, \ldots, m \\ 0 & j = m+1, \ldots, muv \end{cases}$$

where the shape parameters s_j *are defined in Appendix A. The distributions of* W_j, W_{1jk}^* *and* W_{2k}^*, *are defined in* (3.10), (3.12) *and* (3.13).

The case for $v = 1$ is equivalent to the test for block compound symmetric covariance structure and it is addressed in Coelho and Roy (2013).

4. The characteristic function of $W = -\log\Lambda$

The reason why in the previous section we actually obtain two equivalent representations for the exact distribution of Λ, which are the one obtained from (3.8) and the one in Theorem 1 is because the first of these neither does it yield a manageable cumulative distribution function nor is it adequate to lead to a sharp approximation to the exact distribution of Λ.

In order to be able to obtain a very sharp and manageable approximation to the exact distribution of Λ, we will base our developments in the representation given by Theorem 1 and Corollary 1.

From Theorem 1 and expressions (3.1)–(3.4) we may write the characteristic function (c.f.) of $W = -\log\Lambda$ as

$$\begin{aligned}\Phi_W(t) &= E\left(e^{\mathrm{i}tW}\right) = E\left(\Lambda^{-\mathrm{i}t}\right)\\ &= \underbrace{\left\{\prod_{j=2}^{muv}\left(\frac{n-j}{n}\right)^{\mu_j}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-\mu_j}\right\}}_{\Phi_{W,1}(t)} \times \underbrace{\Phi_{a,2}(-\mathrm{i}t)\ \Phi_{b,2}(-\mathrm{i}t)\ \Phi_{c,2}(-\mathrm{i}t)}_{\Phi_{W,2}(t)}\end{aligned} \tag{4.1}$$

where μ_j is given by (3.18), $\Phi_{a,2}(\,\cdot\,)$, $\Phi_{b,2}(\,\cdot\,)$ and $\Phi_{c,2}(\,\cdot\,)$ are defined in (3.1)–(3.4), and $\Phi_{W,1}(t)$ is actually equal to $\Phi_{a,1}(-\mathrm{i}t)\Phi_{b,1}(-\mathrm{i}t)\Phi_{c,1}(-\mathrm{i}t)$.

For $v = 1$ and $v = 2$, according to Corollary 1 in the previous section $\Phi_W(t)$ reduces to

$$\Phi_W(t) = \underbrace{\left\{\prod_{j=2}^{muv}\left(\frac{n-j}{n}\right)^{\mu_j^*}\left(\frac{n-j}{n}-\mathrm{i}t\right)^{-\mu_j^*}\right\}}_{\Phi_{W,1}(t)} \times \underbrace{\Phi_{a,2}(-\mathrm{i}t)\ \Phi_{b,2}(-\mathrm{i}t)}_{\Phi_{W,2}(t)} \tag{4.2}$$

for μ_j^* given by (3.20) and where now $\Phi_{W,1}(t)$ is equal to $\Phi_{a,1}(-\mathrm{i}t)\Phi_{b,1}(-\mathrm{i}t)$.

Expressions (4.1) and (4.2), together with expressions (3.17) and (3.19), show that the exact distribution of $W = -\log\Lambda$ is the same as that of the sum of $muv-1$ independent Gamma random variables with an independent sum of a number of independent Logbeta random variables.

Then, in building the near-exact distributions we will keep $\Phi_{W,1}(t)$ untouched and will approximate $\Phi_{W,2}(t)$ asymptotically by the c.f. of a finite mixture of Gamma distributions.

5. Near-exact distributions

Indeed, based on the result in Section 5 of Tricomi and Erdélyi (1951), we may, for increasing values of a, asymptotically replace the distribution of any $Logbeta(a,b)$ distributed random variable by an infinite mixture of $\Gamma(b+\ell,a)$ distributions ($\ell = 0,1,\dots$). As such, we could replace $\Phi_{W,2}(t)$ in either (4.1) or (4.2) by the c.f. of the sum of infinite mixtures of Gamma distributions, which would be the same as the c.f. of an infinite mixture of sums of Gamma distributions. Although it happens that these Gamma distributions would have different rate parameters, these parameters would anyway be of comparable magnitude. As such, in building our near-exact distributions for $W = -\log\Lambda$ and Λ, while we will leave $\Phi_{W,1}(t)$ unchanged, we will replace $\Phi_{W,2}(t)$ in either (4.1) or (4.2), by

$$\Phi^*(t) = \sum_{\ell=0}^{m^*}\pi_\ell\,\nu^{r+\ell}(\nu-\mathrm{i}t)^{-(r+\ell)} \tag{5.1}$$

which is the c.f. of a finite mixture of $\Gamma(r+\ell,\nu)$ distributions, where, for the general case in (4.1) and for k^* in (3.2) and $v>1$, we will take

$$
\begin{aligned}
r \;=\;& \frac{k^*}{2}+\sum_{j=1}^{\lfloor m/2\rfloor}\sum_{k=1}^{v(u-1)}\frac{k-2j}{v(u-1)}-\left\lfloor\frac{k-2j}{v(u-1)}\right\rfloor+\sum_{j=1}^{\lfloor m/2\rfloor}\sum_{k=1}^{v-1}\frac{k-2j}{v(u-1)}-\left\lfloor\frac{k-2j}{v-1}\right\rfloor\\
&+(m \perp\!\!\!\perp 2)\left(\sum_{k=1}^{v(u-1)}\frac{m-1}{2}+\frac{2k-m-1}{2v(u-1)}-\left\lfloor\frac{m-1}{2}+\frac{2k-m-1}{2v(u-1)}\right\rfloor+\right.\\
&\qquad\left.\sum_{k=1}^{v-1}\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}-\left\lfloor\frac{m-1}{2}+\frac{2k-m-1}{2(v-1)}\right\rfloor\right)\\
=\;& \begin{cases}\dfrac{m}{4}(uv-3) & \text{even } m\\[2mm] \dfrac{1}{2}\left\lfloor\dfrac{uv}{2}\right\rfloor+\dfrac{m+1}{4}(uv-3) & \text{odd } m\,,\end{cases}
\end{aligned}
\tag{5.2}
$$

which is the sum of all the second parameters of the Logbeta distributions in $\Phi_{W,2}(t)$ in (4.1), while for the particular case in (4.2) we will take

$$
\begin{aligned}
r \;=\;& \frac{k^*}{2}+\sum_{j=1}^{\lfloor m/2\rfloor}\sum_{k=1}^{v(u-1)}\frac{k-2j}{v(u-1)}-\left\lfloor\frac{k-2j}{v(u-1)}\right\rfloor\\
&+(m \perp\!\!\!\perp 2)\sum_{k=1}^{v(u-1)}\frac{m-1}{2}+\frac{2k-m-1}{2v(u-1)}-\left\lfloor\frac{m-1}{2}+\frac{2k-m-1}{2v(u-1)}\right\rfloor\\
=\;& \begin{cases}\dfrac{m}{4}\big(v(u-1)-1\big) & \text{even } m\\[2mm] \dfrac{1}{2}\left\lfloor\dfrac{uv}{2}\right\rfloor+\dfrac{m+1}{4}\big(v(u-1)-1\big) & \text{odd } m\,,\end{cases}
\end{aligned}
\tag{5.3}
$$

which is the sum of all the second parameters of the Logbeta distributions in $\Phi_{W,2}(t)$ in (4.2).

The parameter ν in (5.1) is then taken as the rate parameter in

$$\Phi^{**}(t)=\theta\,\nu^{s_1}(\nu-\mathrm{i}t)^{-s_1}+(1-\theta)\nu^{s_2}(\nu-\mathrm{i}t)^{-s_2}$$

where θ, ν, s_1 and s_2 are determined in such a way that

$$\left.\frac{d\Phi_{W,2}(t)}{dt^h}\right|_{t=0}=\left.\frac{d\Phi^{**}(t)}{dt^h}\right|_{t=0}\qquad\text{for}\qquad h=1,\dots,4\,,$$

while the weights π_ℓ ($\ell=0,\dots,m^*-1$) in (5.1) will then be determined in such a way that

$$\left.\frac{d\Phi_{W,2}(t)}{dt^h}\right|_{t=0}=\left.\frac{d\Phi^{*}(t)}{dt^h}\right|_{t=0}\qquad\text{for}\qquad h=1,\dots,m^*\,,$$

with $\pi_{m^*}=1-\sum_{\ell=0}^{m^*-1}\pi_\ell$.

This procedure yields near-exact distributions for W which will match the first m^* exact moments of W and which have c.f.

$$\Phi_{W,1}(t)\Phi^*(t)\,,$$

with $\Phi_{W,1}(t)$ given by (4.1) or (4.2) and $\Phi^*(t)$ by (5.1), where r, given by (5.2) or (5.3) is always either an integer or a half-integer.

As such, the near-exact distributions developed yield, for W, distributions which, for non-integer r, are mixtures, with weights π_ℓ ($\ell=0,\dots,m^*$), of m^*+1 Generalized Near-Integer Gamma (GNIG) distributions of depth muv with integer shape parameters μ_j ($j=2,\dots,muv$) and real shape parameter r, in the general case, or shape parameters μ_j^* ($j=2,\dots,muv$) for the case of $v=1$ or $v=2$, and corresponding rate parameters $(n-j)/n$ ($j=2,\dots,muv$) and ν, and which, for integer r, are similar mixtures but of Generalized Integer Gamma (GIG) distributions, with the same shape and rate parameters (see Coelho (1998, 2004) and Appendix C for further details on the GIG and GNIG distributions and their probability density and cumulative distribution functions).

Using the notation in Appendix C for the probability density and cumulative distribution functions of the GNIG distribution, the near-exact distributions obtained for W, for the general case of $v>2$ and for the case of non-integer r, will have probability density and cumulative distribution functions respectively of the form

$$f_W^*(w)=\sum_{\ell=0}^{m^*}\pi_\ell\, f^{GNIG}\left(w\,\middle|\,\mu_2,\dots,\mu_{muv};r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n};\nu;muv\right)\,,\quad w>0$$

and

$$F_W^*(w)=\sum_{\ell=0}^{m^*}\pi_\ell\, F^{GNIG}\left(w\,\middle|\,\mu_2,\dots,\mu_{muv};r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n};\nu;muv\right)\,,\quad w>0\,,$$

while the near-exact probability density and cumulative distribution functions of Λ are respectively given by

$$f_\Lambda^*(\lambda)=\sum_{\ell=0}^{m^*}\pi_\ell\, f^{GNIG}\left(-\log\lambda\,\middle|\,\mu_2,\dots,\mu_{muv};r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n};\nu;muv\right)\frac{1}{\lambda}\,,\quad 0<\lambda<1$$

and

$$F_\Lambda^*(\lambda)=\sum_{\ell=0}^{m^*}\pi_\ell\left(1-F^{GNIG}\left(-\log\lambda\,\middle|\,\mu_2,\dots,\mu_{muv};r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n};\nu;muv\right)\right)\,,\quad 0<\lambda<1\,.$$

For the case $v=1$ or $v=2$ all we have to do is to replace the shape parameters μ_j given by (3.18) by the shape parameters μ_j^* given by (3.20) and use r given by (5.3), instead of r given by (5.2).

For integer r, all we have to do is to replace the GNIG probability density and cumulative distribution functions by their GIG counterparts (see Appendix C), yielding near-exact distributions for W, for the general case of $v>2$, with probability density and cumulative distribution functions respectively of the form

$$f_W^*(w)=\sum_{\ell=0}^{m^*}\pi_\ell\, f^{GIG}\left(w;\,\mu_2,\dots,\mu_{muv},r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n},\nu;muv\right)\,,\quad w>0$$

and

$$F_W^*(w)=\sum_{\ell=0}^{m^*}\pi_\ell\, F^{GIG}\left(w;\,\mu_2,\dots,\mu_{muv},r+\ell;\frac{n-2}{n},\dots,\frac{n-muv}{n},\nu;muv\right)\,,\quad w>0\,,$$

while the near-exact probability density and cumulative distribution functions of Λ are respectively given by

$$f^*_\Lambda(\lambda) = \sum_{\ell=0}^{m^*} \pi_\ell \, f^{GIG}\left(-\log\lambda\,;\, \mu_2,\ldots,\mu_{muv}, r+\ell; \frac{n-2}{n},\ldots,\frac{n-muv}{n},\nu; muv\right)\frac{1}{\lambda}\,, \quad 0<\lambda<1 \tag{5.4}$$

and

$$F^*_\Lambda(\lambda) = \sum_{\ell=0}^{m^*} \pi_\ell \left(1 - F^{GIG}\left(-\log\lambda\,;\, \mu_2,\ldots,\mu_{muv}, r+\ell; \frac{n-2}{n},\ldots,\frac{n-muv}{n},\nu; muv\right)\right)\,, \quad 0<\lambda<1\,, \tag{5.5}$$

and where, similar to what happens for the case of non-integer r, for the case $v=1$ or $v=2$ we have to replace the shape parameters μ_j given by (3.18) by the shape parameters μ_j^* given by (3.20) and use r given by (5.3), instead of r given by (5.2).

6. Numerical studies

In order to assess the performance of the near-exact distributions developed we will use

$$\Delta = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left| \frac{\Phi_W(t) - \Phi_{W,1}(t)\Phi^*(t)}{t} \right| dt \tag{6.1}$$

with

$$\Delta \geq \max_w \left| F_W(w) - F^*_W(w) \right|\,,$$

as a measure of proximity between the exact and the near-exact distributions, where $\Phi_W(t)$ is the exact c.f. of W in (4.1) or (4.2) and $F_W(\,\cdot\,)$ and $F^*_W(\,\cdot\,)$ represent respectively the exact and near-exact cumulative distribution functions of W, corresponding respectively to $\Phi_W(t)$ and $\Phi_{W,1}(t)\Phi^*(t)$.

In Tables 1-3 we may analyze values of Δ for different combinations of values of m, u and v and different sample sizes. For each combination of values of m, u and v, at least three different sample sizes, that is, values of n, are used, exceeding the total number of variables muv by 2, 30 and 100. For larger combinations of values of m, u and v, some larger values of n are also used to illustrate the asymptotic behavior of the near-exact distributions in what concerns the sample size. For all near-exact distributions, values of m^* equal to 4, 6 and 10 are used, that is, we used for each case near-exact distributions matching 4, 6 and 10 exact moments of W. Smaller values of Δ indicate a closer agreement with the exact distribution and as such, a better performance of the corresponding near-exact distribution.

We may see how the near-exact distributions developed show very sharp approximations to the exact distribution even for very small samples, that is, for sample sizes hardly exceeding the total number of variables involved. Moreover, they also show clear asymptotic behaviors not only for increasing sample sizes, but also for increasing values of m, u and v, with the asymptotic behavior in terms of sample size becoming apparent for larger sample sizes as the values of m, u and v get larger.

Table 1: Values of the measure Δ for the near-exact distributions for $m = 2$

	$m = 2$						
		m^*				m^*	
n	4	6	10	n	4	6	10
		$u = 2,\ v = 2$				$u = 2,\ v = 5$	
10	4.23×10^{-12}	9.00×10^{-15}	2.64×10^{-19}	22	7.80×10^{-17}	8.63×10^{-22}	8.26×10^{-30}
38	1.68×10^{-14}	3.06×10^{-18}	9.23×10^{-25}	50	2.39×10^{-17}	2.04×10^{-22}	1.42×10^{-31}
108	9.17×10^{-17}	1.97×10^{-21}	8.66×10^{-30}	120	4.65×10^{-19}	9.74×10^{-25}	5.38×10^{-35}
		$u = 5,\ v = 2$				$u = 2,\ v = 10$	
22	1.80×10^{-16}	4.11×10^{-21}	1.39×10^{-29}	42	1.73×10^{-19}	1.88×10^{-25}	1.07×10^{-36}
50	6.28×10^{-17}	6.99×10^{-22}	6.21×10^{-31}	70	4.23×10^{-19}	4.80×10^{-25}	2.73×10^{-36}
120	1.30×10^{-18}	2.97×10^{-24}	1.10×10^{-34}	140	3.16×10^{-20}	1.19×10^{-26}	7.31×10^{-39}
		$u = 5,\ v = 5$				$u = 5,\ v = 10$	
52	1.78×10^{-20}	8.05×10^{-27}	5.49×10^{-39}	102	6.18×10^{-24}	1.72×10^{-31}	2.71×10^{-46}
80	6.71×10^{-20}	3.77×10^{-26}	3.56×10^{-38}	130	5.38×10^{-23}	2.67×10^{-30}	1.23×10^{-44}
150	7.77×10^{-21}	1.71×10^{-27}	2.37×10^{-40}	200	2.24×10^{-23}	7.15×10^{-31}	1.29×10^{-45}
250	8.33×10^{-22}	7.33×10^{-29}	1.58×10^{-42}	300	4.93×10^{-24}	8.25×10^{-32}	3.96×10^{-47}
		$u = 10,\ v = 2$				$u = 10,\ v = 5$	
42	1.10×10^{-19}	1.32×10^{-25}	8.00×10^{-37}	102	6.53×10^{-24}	1.80×10^{-31}	2.77×10^{-46}
70	2.73×10^{-19}	3.35×10^{-25}	1.79×10^{-36}	130	5.69×10^{-23}	2.80×10^{-30}	1.26×10^{-44}
140	2.06×10^{-20}	8.25×10^{-27}	4.20×10^{-39}	200	2.37×10^{-23}	7.50×10^{-31}	1.32×10^{-45}
240	1.83×10^{-21}	2.69×10^{-28}	1.74×10^{-41}	300	5.22×10^{-24}	8.66×10^{-32}	4.06×10^{-47}
		$u = 10,\ v = 10$					
202	3.92×10^{-28}	2.22×10^{-36}	1.02×10^{-53}				
230	4.39×10^{-27}	5.35×10^{-35}	9.94×10^{-52}				
300	4.53×10^{-27}	5.48×10^{-35}	8.66×10^{-52}				
400	2.04×10^{-27}	1.82×10^{-35}	1.39×10^{-52}				
500	8.98×10^{-28}	5.88×10^{-36}	2.23×10^{-53}				
1000	4.50×10^{-29}	9.29×10^{-38}	2.99×10^{-56}				

7. A Real Data Example

To illustrate our proposed testing method, we test the hypothesis (2.1) on a real data set. The data set is from Johnson and Wichern (2007, p. 43 and p. 353). An investigator measured the mineral content of bones (radius, humerus and ulna) by photon absorptiometry to examine whether dietary supplements would slow bone loss in 25 older women. Measurements were recorded for the three bones on the dominant and non-dominant sides (Johnson and Wichern, 2007, p. 43). Thus, the data is two-level multivariate and clearly $u = 2$ and $m = 3$.

The bone mineral contents for the first 24 women one year after their participation in an experimental program is given in Johnson and Wichern (2007, p. 353). Thus, for our analysis we take only the first 24 women in the first data set, and combine these two data sets side by side into a new one, which we analyze in this article. Thus, this new data set has a three-level multivariate structure, with $v = 2$, $u = 2$ and $m = 3$.

We rearrange the variables in the new data set by grouping together the mineral content of the dominant sides of radius, humerus and ulna as the first three variables, that is, the variables in the first location ($s = 1$) and then the mineral contents for the non-dominant side of the same bones ($s = 2$) on the first year ($t = 1$) of the experiment; and do the same thing at the second year ($t = 2$) of the experiment.

Table 2: Values of the measure Δ for the near-exact distributions for $m = 5$

$m = 5$

	m^*				m^*		
n	4	6	10	n	4	6	10
		$u=2\,,\quad v=2$				$u=2\,,\quad v=5$	
22	8.71×10^{-16}	1.35×10^{-19}	5.78×10^{-27}	52	7.45×10^{-21}	7.79×10^{-27}	5.53×10^{-38}
50	2.88×10^{-16}	2.22×10^{-20}	1.28×10^{-27}	80	3.98×10^{-20}	1.16×10^{-25}	7.02×10^{-37}
120	5.50×10^{-18}	8.15×10^{-23}	1.98×10^{-31}	150	1.12×10^{-20}	8.20×10^{-27}	1.09×10^{-38}
				250	1.52×10^{-21}	4.06×10^{-28}	9.06×10^{-41}
				350	3.47×10^{-22}	4.79×10^{-29}	3.05×10^{-42}
		$u=5\,,\quad v=2$				$u=2\,,\quad v=10$	
52	2.14×10^{-19}	2.56×10^{-25}	9.77×10^{-38}	102	3.31×10^{-22}	3.53×10^{-29}	4.93×10^{-43}
80	7.41×10^{-19}	1.11×10^{-24}	9.36×10^{-37}	130	2.76×10^{-21}	5.32×10^{-28}	2.25×10^{-41}
150	7.86×10^{-20}	4.64×10^{-26}	1.32×10^{-38}	200	1.09×10^{-21}	1.36×10^{-28}	2.38×10^{-42}
250	8.06×10^{-21}	1.91×10^{-27}	1.04×10^{-40}	300	2.31×10^{-22}	1.53×10^{-29}	7.36×10^{-44}
350	1.66×10^{-21}	2.09×10^{-28}	3.40×10^{-42}	500	2.43×10^{-23}	6.50×10^{-31}	5.05×10^{-46}
		$u=5\,,\quad v=5$				$u=5\,,\quad v=10$	
127	5.22×10^{-23}	2.70×10^{-30}	8.95×10^{-45}	252	4.03×10^{-26}	1.58×10^{-34}	2.67×10^{-51}
155	5.10×10^{-22}	5.07×10^{-29}	5.83×10^{-43}	280	4.72×10^{-25}	3.93×10^{-33}	2.85×10^{-49}
225	2.91×10^{-22}	2.16×10^{-29}	1.35×10^{-43}	350	6.35×10^{-25}	5.45×10^{-33}	4.19×10^{-49}
450	2.07×10^{-23}	5.13×10^{-31}	3.55×10^{-46}	450	3.70×10^{-25}	2.46×10^{-33}	1.13×10^{-49}
650	3.99×10^{-24}	5.10×10^{-32}	9.27×10^{-48}	1000	1.70×10^{-26}	3.15×10^{-35}	1.11×10^{-52}
		$u=10\,,\quad v=2$				$u=10\,,\quad v=5$	
102	3.18×10^{-22}	3.16×10^{-29}	4.45×10^{-43}	252	4.41×10^{-26}	1.76×10^{-34}	3.14×10^{-51}
130	2.70×10^{-21}	4.83×10^{-28}	2.05×10^{-41}	280	5.16×10^{-25}	4.38×10^{-33}	3.34×10^{-49}
200	1.09×10^{-21}	1.26×10^{-28}	2.21×10^{-42}	350	6.94×10^{-25}	6.09×10^{-33}	4.92×10^{-49}
500	2.49×10^{-23}	6.13×10^{-31}	4.78×10^{-46}	450	4.05×10^{-25}	2.75×10^{-33}	1.32×10^{-49}
				1000	1.86×10^{-26}	3.52×10^{-35}	1.30×10^{-52}
		$u=10\,,\quad v=10$					
502	3.22×10^{-29}	1.01×10^{-38}	9.76×10^{-58}				
530	3.52×10^{-28}	2.35×10^{-37}	9.82×10^{-56}				
600	7.50×10^{-28}	6.22×10^{-37}	3.97×10^{-55}				
1000	3.27×10^{-28}	1.76×10^{-37}	4.67×10^{-56}				
5000	3.02×10^{-31}	9.42×10^{-42}	8.31×10^{-63}				

The resulting m.l.e. of $\mathbf{\Sigma}$ is

$$\boldsymbol{A}^{+}=\left(\begin{array}{ccc|ccc|ccc|ccc}
0.01278 & 0.02180 & 0.00861 & 0.01001 & 0.01967 & 0.00742 & 0.01271 & 0.02461 & 0.00840 & 0.01019 & 0.01937 & 0.00820\\
0.02180 & 0.07898 & 0.01556 & 0.01762 & 0.06574 & 0.01150 & 0.02469 & 0.08474 & 0.01837 & 0.01868 & 0.06899 & 0.01424\\
0.00861 & 0.01556 & 0.01039 & 0.00769 & 0.01678 & 0.00683 & 0.00820 & 0.02036 & 0.01026 & 0.00834 & 0.01713 & 0.00698\\
\hline
0.01001 & 0.01762 & 0.00769 & 0.01082 & 0.02041 & 0.00802 & 0.01025 & 0.02081 & 0.00777 & 0.01068 & 0.02072 & 0.00889\\
0.01967 & 0.06574 & 0.01678 & 0.02041 & 0.06870 & 0.01578 & 0.02212 & 0.07380 & 0.01974 & 0.02184 & 0.07289 & 0.01869\\
0.00742 & 0.01150 & 0.00683 & 0.00802 & 0.01578 & 0.00928 & 0.00823 & 0.01540 & 0.00678 & 0.00853 & 0.01630 & 0.00960\\
\hline
0.01271 & 0.02469 & 0.00820 & 0.01025 & 0.02212 & 0.00823 & 0.01487 & 0.02827 & 0.00860 & 0.01070 & 0.02315 & 0.00890\\
0.02461 & 0.08474 & 0.02036 & 0.02081 & 0.07380 & 0.01540 & 0.02827 & 0.10068 & 0.02296 & 0.02251 & 0.08056 & 0.01756\\
0.00840 & 0.01837 & 0.01026 & 0.00777 & 0.01974 & 0.00678 & 0.00860 & 0.02296 & 0.01103 & 0.00832 & 0.02033 & 0.00723\\
\hline
0.01019 & 0.01868 & 0.00834 & 0.01068 & 0.02184 & 0.00853 & 0.01070 & 0.02251 & 0.00832 & 0.01127 & 0.02259 & 0.00928\\
0.01937 & 0.06899 & 0.01713 & 0.02072 & 0.07289 & 0.01630 & 0.02315 & 0.08056 & 0.02033 & 0.02259 & 0.08083 & 0.01884\\
0.00820 & 0.01424 & 0.00698 & 0.00889 & 0.01869 & 0.00960 & 0.00890 & 0.01756 & 0.00723 & 0.00928 & 0.01884 & 0.01202
\end{array}\right).$$

Table 3: Values of the measure Δ for the near-exact distributions for $m=10$

		$m=10$					
		m^*				m^*	
n	4	6	10	n	4	6	10
		$u=2\,,\quad v=2$				$u=2\,,\quad v=5$	
42	$1.43{\times}10^{-17}$	$1.45{\times}10^{-22}$	$3.20{\times}10^{-32}$	102	$1.30{\times}10^{-23}$	$2.68{\times}10^{-31}$	$1.27{\times}10^{-46}$
70	$2.03{\times}10^{-17}$	$2.06{\times}10^{-22}$	$4.03{\times}10^{-32}$	130	$8.59{\times}10^{-22}$	$1.96{\times}10^{-28}$	$1.48{\times}10^{-41}$
140	$7.01{\times}10^{-19}$	$2.26{\times}10^{-24}$	$4.43{\times}10^{-35}$	200	$2.98{\times}10^{-22}$	$4.60{\times}10^{-29}$	$1.49{\times}10^{-42}$
				300	$5.74{\times}10^{-23}$	$4.83{\times}10^{-30}$	$4.44{\times}10^{-44}$
				500	$5.53{\times}10^{-24}$	$1.94{\times}10^{-31}$	$2.95{\times}10^{-46}$
				750	$7.93{\times}10^{-25}$	$1.33{\times}10^{-32}$	$4.45{\times}10^{-48}$
				1000	$1.95{\times}10^{-25}$	$1.90{\times}10^{-33}$	$2.13{\times}10^{-49}$
		$u=5\,,\quad v=2$				$u=2\,,\quad v=10$	
102	$2.49{\times}10^{-24}$	$3.43{\times}10^{-31}$	$2.19{\times}10^{-45}$	202	$2.19{\times}10^{-26}$	$5.75{\times}10^{-35}$	$3.99{\times}10^{-52}$
130	$1.57{\times}10^{-23}$	$5.03{\times}10^{-30}$	$1.02{\times}10^{-43}$	230	$2.57{\times}10^{-25}$	$1.40{\times}10^{-33}$	$4.03{\times}10^{-50}$
200	$3.21{\times}10^{-24}$	$1.21{\times}10^{-30}$	$1.09{\times}10^{-44}$	300	$2.86{\times}10^{-25}$	$1.46{\times}10^{-33}$	$3.73{\times}10^{-50}$
300	$2.45{\times}10^{-25}$	$1.30{\times}10^{-31}$	$3.41{\times}10^{-46}$	500	$6.25{\times}10^{-26}$	$1.61{\times}10^{-34}$	$1.04{\times}10^{-51}$
500	$1.37{\times}10^{-26}$	$5.31{\times}10^{-33}$	$2.34{\times}10^{-48}$	1000	$3.37{\times}10^{-27}$	$2.58{\times}10^{-36}$	$1.49{\times}10^{-54}$
		$u=5\,,\quad v=5$				$u=5\,,\quad v=10$	
252	$6.13{\times}10^{-26}$	$2.94{\times}10^{-34}$	$9.95{\times}10^{-51}$	502	$5.33{\times}10^{-30}$	$7.80{\times}10^{-40}$	$1.50{\times}10^{-59}$
280	$7.19{\times}10^{-25}$	$7.31{\times}10^{-33}$	$1.06{\times}10^{-48}$	530	$5.84{\times}10^{-29}$	$1.81{\times}10^{-38}$	$1.51{\times}10^{-57}$
350	$9.71{\times}10^{-25}$	$1.02{\times}10^{-32}$	$1.56{\times}10^{-48}$	600	$1.25{\times}10^{-28}$	$4.80{\times}10^{-38}$	$6.12{\times}10^{-57}$
750	$8.94{\times}10^{-26}$	$3.32{\times}10^{-34}$	$6.30{\times}10^{-51}$	1000	$5.49{\times}10^{-29}$	$1.37{\times}10^{-38}$	$7.25{\times}10^{-58}$
1500	$4.19{\times}10^{-27}$	$4.47{\times}10^{-36}$	$6.99{\times}10^{-54}$	2000	$3.55{\times}10^{-30}$	$2.85{\times}10^{-40}$	$1.55{\times}10^{-60}$
		$u=10\,,\quad v=2$				$u=10\,,\quad v=5$	
202	$1.20{\times}10^{-26}$	$3.09{\times}10^{-35}$	$1.91{\times}10^{-52}$	502	$4.91{\times}10^{-29}$	$1.64{\times}10^{-38}$	$2.66{\times}10^{-57}$
230	$1.37{\times}10^{-25}$	$7.41{\times}10^{-34}$	$1.90{\times}10^{-50}$	530	$5.38{\times}10^{-28}$	$3.81{\times}10^{-37}$	$2.67{\times}10^{-55}$
300	$1.47{\times}10^{-25}$	$7.51{\times}10^{-34}$	$1.72{\times}10^{-50}$	600	$1.15{\times}10^{-27}$	$1.01{\times}10^{-36}$	$1.08{\times}10^{-54}$
500	$3.03{\times}10^{-26}$	$7.93{\times}10^{-35}$	$4.68{\times}10^{-52}$	1000	$5.08{\times}10^{-28}$	$2.87{\times}10^{-37}$	$1.27{\times}10^{-55}$
1000	$1.57{\times}10^{-27}$	$1.24{\times}10^{-36}$	$6.54{\times}10^{-55}$	2000	$3.30{\times}10^{-29}$	$5.97{\times}10^{-39}$	$2.70{\times}10^{-58}$
				5000	$4.77{\times}10^{-31}$	$1.55{\times}10^{-41}$	$2.26{\times}10^{-62}$
		$u=10\,,\quad v=10$					
1002	$4.36{\times}10^{-33}$	$4.29{\times}10^{-44}$	$4.19{\times}10^{-66}$				
1030	$4.03{\times}10^{-32}$	$8.04{\times}10^{-43}$	$3.14{\times}10^{-64}$				
1100	$1.03{\times}10^{-31}$	$2.74{\times}10^{-42}$	$1.92{\times}10^{-63}$				
2000	$6.59{\times}10^{-32}$	$1.25{\times}10^{-42}$	$4.37{\times}10^{-64}$				
5000	$1.60{\times}10^{-33}$	$6.46{\times}10^{-45}$	$1.03{\times}10^{-67}$				

We see that the variance-covariance matrices ($\boldsymbol{U}_0$) of the three mineral contents for the dominant and non-dominant sides appear very similar for the first as well as for the second years. Also, the covariance matrices ($\boldsymbol{U}_1$) of the three bones between the dominant and non-dominant sides seem to be fairly similar for both the years. Finally, the covariance matrices ($\boldsymbol{W}$) of the three bones between the two years seem to be similar too.

Thus, we may think about testing the hypothesis that the population covariance matrix has a doubly exchangeable covariance structure. As stated in Section 2, this is equivalent to test the hypothesis in (2.2).

We thus compute the m.l.e. of $\boldsymbol{\Sigma}^*$, that is the matrix

$$
\begin{aligned}
\boldsymbol{A} &= \boldsymbol{\Gamma}^{\bullet}\boldsymbol{\Gamma}^{*}\boldsymbol{A}^{+}\boldsymbol{\Gamma}^{*\prime}\boldsymbol{\Gamma}^{\bullet\prime} \\
&= \left(\begin{array}{ccc:ccc|ccc:ccc}
0.04470 & 0.08722 & 0.03335 & 0.00241 & 0.00228 & -0.00038 & -0.00098 & -0.00380 & -0.00083 & -0.00044 & -0.00291 & 0.00070 \\
0.08722 & 0.30566 & 0.06989 & 0.00462 & 0.01345 & 0.00573 & -0.00470 & -0.01587 & -0.00548 & -0.00168 & -0.00480 & -0.00031 \\
0.03335 & 0.06989 & 0.03452 & -0.00007 & -0.00191 & 0.00036 & -0.00085 & -0.00455 & -0.00105 & 0.00020 & -0.00375 & 0.00062 \\
\hdashline
0.00241 & 0.00462 & -0.00007 & 0.00356 & 0.00525 & 0.00092 & -0.00038 & 0.00024 & 0.00001 & -0.00029 & 0.00007 & -0.00008 \\
0.00228 & 0.01345 & -0.00191 & 0.00525 & 0.01657 & 0.00355 & 0.00006 & 0.00001 & 0.00032 & -0.00022 & -0.00105 & 0.00005 \\
-0.00038 & 0.00573 & 0.00036 & 0.00092 & 0.00355 & 0.00670 & 0.00036 & -0.00061 & 0.00043 & 0.00050 & -0.00047 & -0.00064 \\
\hline
-0.00098 & -0.00470 & -0.00085 & -0.00038 & 0.00006 & 0.00036 & 0.00087 & 0.00079 & 0.00007 & 0.00037 & -0.00009 & 0.00019 \\
-0.00380 & -0.01587 & -0.00455 & 0.00024 & 0.00001 & -0.00061 & 0.00079 & 0.00523 & -0.00023 & 0.00025 & 0.00161 & 0.00025 \\
-0.00083 & -0.00548 & -0.00105 & 0.00001 & 0.00032 & 0.00043 & 0.00007 & -0.00023 & 0.00090 & 0.00018 & -0.00016 & -0.00030 \\
\hdashline
-0.00044 & -0.00168 & 0.00020 & -0.00029 & -0.00022 & 0.00050 & 0.00037 & 0.00025 & 0.00018 & 0.00060 & -0.00019 & 0.00018 \\
-0.00291 & -0.00480 & -0.00375 & 0.00007 & -0.00105 & -0.00047 & -0.00009 & 0.00161 & -0.00016 & -0.00019 & 0.00173 & -0.00006 \\
0.00070 & -0.00031 & 0.00062 & -0.00008 & 0.00005 & -0.00064 & 0.00019 & 0.00025 & -0.00030 & 0.00018 & -0.00006 & 0.00060
\end{array}\right),
\end{aligned}
$$

where the orthogonal matrices $\boldsymbol{\Gamma}^{\bullet}$ and $\boldsymbol{\Gamma}^{*}$ are

$$
\boldsymbol{\Gamma}^{\bullet} = \boldsymbol{I}_v \otimes (\underset{u\times u}{\boldsymbol{C}^{*\prime}} \otimes \boldsymbol{I}_m) = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix},
$$

and

$$
\boldsymbol{\Gamma}^{*} = \underset{v\times v}{\boldsymbol{C}^{\prime}} \otimes \boldsymbol{I}_{mu} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix},
$$

with

$$
\boldsymbol{C} = \boldsymbol{C}^{*} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
$$

Now, from (2.9) we have

$$\Lambda = \left(2^6 \, \frac{|\boldsymbol{A}|}{|\boldsymbol{A}_1| \, |\boldsymbol{A}_2 + \boldsymbol{A}_4| \, |\boldsymbol{A}_3|} \right)^{n/2} ,$$

where $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_4$ denote the four diagonal blocks of dimension $3{\times}3$ of $\boldsymbol{A}$ and where $n = 24$. The calculated value of Λ is $\lambda = 6.18767{\times}10^{-35}$.

Using then the near-exact distribution for Λ which matches $m^* = 4$ exact moments, with probability density and cumulative distribution functions given by (5.4) and (5.5), with $r = 2$, as given by (5.3), and the other shape parameters μ_j^* $(j = 2, \ldots, muv = 12)$ given by (3.20), with

$$\begin{array}{rcl} r_j^+ & = & \{0, 1, 5, 4, 4, 3, 3, 2, 2, 1, 1\} \\ s_j^+ & = & \{1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \\ \mu_j^* & = & \{1, 2, 5, 4, 4, 3, 3, 2, 2, 1, 1\} \; = \; r_j^+ + s_j^+ \, , \end{array}$$

we obtain a near-exact p-value of 0.000039. Thus, we should reject the null hypothesis that the covariance structure is of the doubly exchangeable type, even though at first sight this appeared to be a plausible hypothesis.

In case we had used the common chi-square approximation for the distribution of l.r.t. statistics, we would have $-2 \log \Lambda \overset{a}{\sim} \chi^2_{\{uvm(uvm+1)/2\} - \{3m(m+1)/2\}} \equiv \chi^2_{60}$, which would give a p-value of $1.08083{\times}10^{-10}$. Although for common values of the level α this p-value would lead to the same decision, in terms of the rejection of the null hypothesis, it also shows that the chi-square approximation really yields p-values which are far away from the exact value. This indeed happens even for quite large sample sizes. In case this fact may not seem of that much great significance, we should note that while the near-exact distribution that we used to compute the near-exact p-value yields a value of $2.32{\times}10^{-13}$ for the measure Δ in (6.1), the chi-square approximation yields for this same measure the value of $5.59{\times}10^{-02}$. Furthermore, while for a computed value of Λ of $2.067{\times}10^{-24}$ the near-exact distribution yields a p-value which rounded to five decimal places is equal to 0.05000, the chi-square approximation yields a p-value of 0.00011, which clearly shows that the common chi-square approximation, opposite to the near-exact approach, leads, for the classical testing approach, to too many rejections of the null hypothesis, or, equivalently, in general, to too low p-values, clearly inadequate for any practical purposes.

In Figure 1 we may analyze the plots of the cumulative distribution function for the near-exact distribution for $-\log \Lambda$ and for the $\Gamma(30, 1)$ distribution which corresponds to the χ^2_{60} approximation for $-2 \log \Lambda$, to see how much they differ.

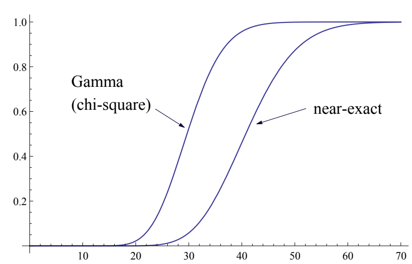

Figure 1: Plots of the cumulative distribution function for the near-exact distribution for $-\log \Lambda$ and for the $\Gamma(30, 1)$ distribution which corresponds to the χ^2_{60} approximation for $-2 \log \Lambda$

8. Conclusions

We may see how the techniques used to handle the null hypothesis in (2.1), by first bringing it to the form in (2.2) and then using the decomposition in (2.3) enabled the development of very accurate near-exact distributions for the l.r.t. statistic.

From the results of the numerical studies carried out we see that the near-exact distributions developed show an interesting set of nice features. They not only have a good asymptotic behavior for increasing sample sizes, but also an extraordinary performance for very small sample sizes, as for example for sample sizes exceeding only by 2 the overall number of variables. Furthermore, these near-exact distributions also display a marked asymptotic behavior for increasing values of m, u and v. All these features add up to make the near-exact approximations developed the best choice for practical applications of the test studied.

Moreover, given the results in Sections 9.11 and 10.11 of Anderson (2003), the results presented concerning the exact distribution of the l.r.t. statistic as well the near-exact distributions developed may be made extensive to cases where the vector $\boldsymbol{y}$ has an elliptically contoured distribution.

For $v = 1$, the present test reduces to the test for block compound symmetry studied in Coelho and Roy (2013) and for $v = 1$ and $m = 1$ to the equivariance-equicorrelation Wilks (1946) test.

Acknowledgments

This research was supported by CMA/FCT/UNL, under the project PEst-OE/MAT/ UI0297/ 2011.

The second author would like to thank the College of Business at the University of Texas at San Antonio for a summer grant which was determinant for the development of the present research.

Appendix A. Expressions for the shape parameters in the moment expressions for Λ_b and Λ_c

The shape parameters s_j in (3.3) are given by

$$s_j = \begin{cases} s^*_{j-1} & \text{for } j=2,\dots,m, \\ & \text{except for } j=m-2\alpha_1 \\ s^*_{j-1}+(m \perp\!\!\!\perp 2)(\alpha_2-\alpha_1)\left(v(u-1)-\frac{m-1}{2}+v(u-1)\left\lfloor\frac{m}{2v(u-1)}\right\rfloor\right) & \text{for } j=m-2\alpha_1 \end{cases}$$

with

$$s^*_j = \begin{cases} \gamma_j & \text{for} \quad j=1,\dots,\alpha+1 \\ v(u-1)\left(\left\lfloor\frac{m}{2}\right\rfloor-\left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=\alpha+2,\dots,\min(m-2\alpha_1,m-1) \\ & \text{and} \quad j=2+m-2\alpha_1,\dots,2\left\lfloor\frac{m}{2}\right\rfloor-1, \text{ by steps of } 2 \\ v(u-1)\left(\left\lfloor\frac{m+1}{2}\right\rfloor-\left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=1+m-2\alpha_1,\dots,m-1, \text{ by steps of } 2\,, \end{cases}$$

and

$$\alpha=\left\lfloor\frac{m-1}{v(u-1)}\right\rfloor\,,\qquad \alpha_1=\left\lfloor\frac{v(u-1)-1}{v(u-1)}\,\frac{m-1}{2}\right\rfloor\,,\qquad \alpha_2=\left\lfloor\frac{v(u-1)-1}{v(u-1)}\,\frac{m+1}{2}\right\rfloor\,,$$

where, for $j=1,\dots,\alpha$,

$$\gamma_j=\left\lfloor\frac{v(u-1)}{2}\right\rfloor\left((j-1)v(u-1)-2\left((v(u-1)+1)\perp\!\!\!\perp 2\right)\left\lfloor\frac{j}{2}\right\rfloor\right)+\left\lfloor\frac{v(u-1)}{2}\right\rfloor\left\lfloor\frac{v(u-1)+j\perp\!\!\!\perp 2}{2}\right\rfloor$$

and

$$\gamma_{\alpha+1}=-\left(\left\lfloor\frac{m}{2}\right\rfloor-\alpha\left\lfloor\frac{v(u-1)}{2}\right\rfloor\right)^2+v(u-1)\left(\left\lfloor\frac{m}{2}\right\rfloor-\left\lfloor\frac{\alpha+1}{2}\right\rfloor\right)$$
$$+(v(u-1)\perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{m}{2}\right\rfloor+\frac{\alpha\perp\!\!\!\perp 2}{4}-\frac{\alpha^2}{4}-\alpha^2\left\lfloor\frac{v(u-1)}{2}\right\rfloor\right)\,.$$

The shape parameters δ_j in (3.4) are given by

$$\delta_j = \begin{cases} \delta^*_{j-1} & \text{for } j=2,\dots,m, \\ & \text{except for } j=m-2\alpha_1 \\ \delta^*_{j-1}+(m \perp\!\!\!\perp 2)(\alpha_2-\alpha_1)\left((v-1)-\frac{m-1}{2}+(v-1)\left\lfloor\frac{m}{2(v-1)}\right\rfloor\right) & \text{for } j=m-2\alpha_1 \end{cases}$$

with

$$\delta^*_j = \begin{cases} \gamma_j & \text{for} \quad j=1,\dots,\alpha+1 \\ (v-1)\left(\left\lfloor\frac{m}{2}\right\rfloor-\left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=\alpha+2,\dots,\min(m-2\alpha_1,m-1) \\ & \text{and} \quad j=2+m-2\alpha_1,\dots,2\left\lfloor\frac{m}{2}\right\rfloor-1, \text{ by steps of } 2 \\ (v-1)\left(\left\lfloor\frac{m+1}{2}\right\rfloor-\left\lfloor\frac{j}{2}\right\rfloor\right) & \text{for} \quad j=1+m-2\alpha_1,\dots,m-1, \text{ by steps of } 2\,, \end{cases}$$

and

$$\alpha=\left\lfloor\frac{m-1}{v-1}\right\rfloor\,,\qquad \alpha_1=\left\lfloor\frac{v-2}{v-1}\,\frac{m-1}{2}\right\rfloor\,,\qquad \alpha_2=\left\lfloor\frac{v-2}{v-1}\,\frac{m+1}{2}\right\rfloor\,,$$

where, for $j=1,\dots,\alpha$,

$$\gamma_j=\left\lfloor\frac{v-1}{2}\right\rfloor\left((j-1)(v-1)-2\,(v\perp\!\!\!\perp 2)\left\lfloor\frac{j}{2}\right\rfloor\right)+\left\lfloor\frac{v-1}{2}\right\rfloor\left\lfloor\frac{v-1+j\perp\!\!\!\perp 2}{2}\right\rfloor$$

and

$$\gamma_{\alpha+1} = -\left(\left\lfloor\frac{m}{2}\right\rfloor - \alpha\left\lfloor\frac{v-1}{2}\right\rfloor\right)^2 + (v-1)\left(\left\lfloor\frac{m}{2}\right\rfloor - \left\lfloor\frac{\alpha+1}{2}\right\rfloor\right)$$
$$+((v-1)\perp\!\!\!\perp 2)\left(\alpha\left\lfloor\frac{m}{2}\right\rfloor + \frac{\alpha\perp\!\!\!\perp 2}{4} - \frac{\alpha^2}{4} - \alpha^2\left\lfloor\frac{v-1}{2}\right\rfloor\right).$$

Appendix B. Gamma distribution and some related results

We say that the random variable X follows a Gamma distribution with shape parameter $r > 0$ and rate parameter $\lambda > 0$, if the probability density function of X is

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1}, \qquad (x > 0)$$

and we will denote this fact by $X \sim \Gamma(r, \lambda)$. Then we know that the moment generating function of X is

$$M_X(t) = \lambda^r(\lambda - t)^{-r},$$

so that if we define $Z = e^{-X}$ we will have

$$E(Z^h) = E\left(e^{-hX}\right) = M_X(-h) = \lambda^r(\lambda + h)^{-r}.$$

Appendix C. The GIG and GNIG distributions

We will say that a r.v. Y has a GIG (Generalized Integer Gamma) distribution of depth p, with integer shape parameters r_j and rate parameters λ_j $(j = 1, \ldots, p)$, if

$$Y = \sum_{j=1}^{p} Y_j$$

where

$$Y_j \sim \Gamma(r_j, \lambda_j), \qquad r_j \in \mathbb{N},\ \lambda_j > 0,\ j = 1, \ldots, p$$

are p independent integer Gamma or Erlang r.v.'s, with $\lambda_j \neq \lambda_{j'}$ for all $j \neq j'$, with $j, j' \in \{1, \ldots, p\}$ Coelho (1998).

The r.v. Y has probability density and cumulative distribution functions given by (see Coelho (1998)),

$$f^{GIG}(y; r_1, \ldots, r_p; \lambda_1, \ldots, \lambda_p; p) = K \sum_{j=1}^{p} P_j(y)\, e^{-\lambda_j y}, \qquad (y > 0)$$

and

$$F^{GIG}(y; r_1, \ldots, r_p; \lambda_1, \ldots, \lambda_p; p) = 1 - K \sum_{j=1}^{p} P_j^*(y)\, e^{-\lambda_j y}, \qquad (y > 0)$$

where $K = \prod_{j=1}^{p} \lambda_j^{r_j}$,

$$P_j(y) = \sum_{k=1}^{r_j} c_{j,k}\, y^{k-1}, \quad P_j^*(y) = \sum_{k=1}^{r_j} c_{j,k} \sum_{i=0}^{k-1} \frac{y^i\,(k-1)!}{i!\,\lambda_j^{k-i}}$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!} \prod_{i=1,i\neq j}^{p} (\lambda_i - \lambda_j)^{-r_i} \ , \qquad j=1,\dots,p \, ,$$

and, for $k=1,\dots,r_j-1$; $j=1,\dots,p$,

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j-k+i-1)!}{(r_j-k-1)!} \, R(i,j,p) \, c_{j,r_j-(k-i)} \, ,$$

where

$$R(i,j,p) = \sum_{k=1,k\neq j}^{p} r_k \, (\lambda_j-\lambda_k)^{-i} \quad (i=1,\dots,r_j-1) \, .$$

If Y_p has a Gamma distribution with a non-integer shape parameter r_p, then we will say that the r.v. Y has a GNIG (Generalized Near-Integer Gamma) distribution of depth p. The probability density and cumulative distribution functions of Y are, for $y>0$, respectively given by Coelho (2004)

$$f^{GNIG}(\, y \mid r_1,\dots,r_{p-1}; r_p; \, \lambda_1,\dots,\lambda_{p-1}; \lambda_p; p) = K \lambda_p^{r_p} \sum_{j=1}^{p-1} e^{-\lambda_j y}$$

$$\times \sum_{k=1}^{r_j} \left\{ c_{j,k} \frac{\Gamma(k)}{\Gamma(k+r)} y^{k+r_p-1} \, {}_1F_1(r_p, k+r_p, -(\lambda_p-\lambda_j)y) \right\} ,$$

and

$$F^{GNIG}(\, y \mid r_1,\dots,r_{p-1}; r_p; \, \lambda_1,\dots,\lambda_{p-1}; \lambda_p; p) = \frac{\lambda_p^{r_p} \, z^{r_p}}{\Gamma(r_p+1)} {}_1F_1(r_p, r_p+1, -\lambda_p z)$$

$$-K\lambda^r \sum_{j=1}^{p-1} e^{-\lambda_j y} \sum_{k=1}^{r_j} \frac{c_{j,k}\Gamma(k)}{\lambda_j^k} \sum_{i=0}^{k-1} \frac{z^{r_p+i}\lambda_j^i}{\Gamma(r_p+1+i)} {}_1F_1(r_p, r_p+1+i, -(\lambda_p-\lambda_j)y) \, ,$$

with $K = \prod_{j=1}^{p-1} \lambda_j^{r_j}$.

References

Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed., Wiley, New Jersey.

Coelho, C. A. (1998). The Generalized Integer Gamma distribution – a basis for distributions in Multivariate Stastistics, J. Multivariate Anal., **64**, 86–102.

Coelho, C. A. (2004). The Generalized Near-Integer Gamma distribution: a basis for near-exact approximations to the distributions of statistics which are the product of an odd number of independent Beta random variables, J. Multivariate Anal., **89**, 191–218.

Coelho, C. A., Marques, F. J. (2012). Near-exact distributions for the likelihood ratio test statistic to test equality of several variance-covariance matrices in elliptically contoured distributions. *Comput. Statist.*, **27**, 627-659.

Coelho, C. A., Marques, F. J. (2012). The multi-sample block-scalar sphericity test — exact and near-exact distributions for its likelihood ratio test statistic. *Comm. Statist. Theory Methods* (in print).

C.A. Coelho, A. Roy, Testing the hypothesis of a block compound symmetric covariance matrix, Working Paper No. 0012MSS-253-2013, College of Business, University of Texas at San Antonio (url: http://business.utsa.edu/wps/MSS/0012MSS-253%20-2013.pdf) (submitted for publication).

Coelho, C. A., Arnold, B. C., Marques, F. J. (2010). Near-exact distributions for certain likelihood ratio test statistics. *J. Stat. Theory Pract.*, **4**, 711-725 (invited paper for the special memorial issue in honor of H. C. Gupta, guest-edited by C. R. Rao).

Johnson, R. A., Wichern, D. W. (2007). *Applied Multivariate Statistical Analysis*, 6th ed., Prentice Hall, New Jersey.

Leiva, R. Roy, A. (2012). Linear discrimination for three-level multivariate data with separable additive mean vector and doubly exchangeable covariance structure, *Computational Statistics and Data Analysis*, **56**, 1644-1661.

Marques, F. J., Coelho, C. A., Arnold, B. C. (2011). A general near-exact distribution theory for the most common likelihood ratio test statistics used in Multivariate Analysis. *Test*, **20**, 180–203.

Marques, F. J., Coelho, C. A. (2012). Near-exact distributions for the likelihood ratio test statistic of the multi-sample block-matrix sphericity test. *Appl. Math. Comput.*, **219**, 2861-2874.

Roy, A., Fonseca, M. (2012). Linear models with doubly exchangeable distributed errors, *Comm. Statist. Theory Methods*, **41**, 2545–2569.

Roy, A., Leiva, R. (2007). Discrimination with Jointly Equicorrelated Multi-level Multivariate Data, *Adv. in Data Analysis and Classif.*, **1**, 175-199.

Roy, A. and Leiva, R. Likelihood Ratio Tests for Triply Multivariate Data with Structured Correlation on Spatial Repeated Measurements. Statistics & Probability Letters, 78(13), 1971-1980 (2008).

F. G. Tricomi, A. Erdélyi, The asymptotic expansion of a ratio of Gamma functions. *Pacific J. Math.* **1**, 133-142 (1951).

S. S. Wilks, Sample criteria for testing equality of means, equality of variances, and equality of covariances in a Normal multivariate distribution. *Ann. Math. Statist.* **17**, 257-281 (1946).